

92-
A
DEMONSTRATION

Of some of the

PRINCIPAL SECTIONS

O F

Sir ISAAC NEWTON's

PRINCIPLES

O F

Natural Philosophy.

IN WHICH

His peculiar METHOD of treating that
useful Subject, is explained, and applied
to some of the chief Phænomena of the
System of the World.

By JOHN CLARKE, D. D.
Dean of Sarum.

L O N D O N:

Printed for JAMES and JOHN KNAPTON,
at the Crown in St. Paul's Church-yard.

MDCCXXX.

17



6 H L 9



THE PREFACE.



THE Principles of natural Philosophy, wrote by the incomparable Sir *Isaac Newton*, must be acknowledged by all Persons that have any Skill in that Science, to be the most compleat Piece that ever was wrote upon the Subject. His principal Design is to explain the Frame and Constitution of Nature, or the present System of the World;

not by Conjectures and Hypotheses, as the antient Philosophers for the most part did, but by strictly examining the Laws of Nature, and finding out by repeated Experiments what they really are. And in order hereunto, He in the first two Books treats of the Principles of Philosophy in a strict mathematical Way, that is, He demonstrates in general the Laws and Qualities of such Motions and Forces as relate principally to Philosophy, and illustrates what he says with several *Scholiums*, which are the Foundation of it, and in which He explains the Density and Resistance of Bodies, the Nature of Space void of such Bodies, and the Motion of Light and Sound. To express the Proportions required in these Particulars, takes up a vast Compass of mathematical Knowledge, more than the Generality of Readers who have studied that Science are capable of understanding, or at least more than
2 they

The PREFACE

v

they would be willing to spend their Time in, before they come to the Application or Use of any of them; wherefore the judicious Author, in the *Preface* to the Third Book, advises the Reader, *not to go over all the Propositions contained in the two First Books, but carefully to read over the Definitions, the Laws of Motion, and the three first Sections of the First Book; and after this to pass to the Third Book, which treats of the System of the World, and then to consult the remaining Propositions of the foregoing Books at Pleasure, as they are there cited.* It is these Three Sections that I have endeavoured to explain in the following Treatise, by giving a particular Demonstration of every Proposition and its Corollaries, and I have added some of the most useful Propositions out of some other Sections, as those of the *direct Descent of Bodies, of the Motions of*

The PREFACE.

Bodies in Pendulums, and of the Motion of Bodies mutually attracting each other, as being some of those which the Author advises to be occasionally consulted. In the Demonstrations I thought it sufficient barely to refer the Reader to the Elements of Euclid (except where the Proposition is an uncommon one, and then it is particularly specified,) lest they should be too much interrupted by perpetual References, and because I suppose every Person to be so much acquainted with the Elements of Geometry as is necessary to qualify him for the Study of Natural Philosophy, otherwise it is like a Man's attempting to compose a Tune before he is able to distinguish the Notes. But all other Propositions with respect to Conick Sections, or any other Curves, are always particularly referred to, that the Reader may have Recourse to them as he sees fit; and I have
made

The P R E F A C E.

vij

made use of Mr. *Milnes's* Book principally, because his Demonstrations are geometrical and very clear, and because in composing it, he had the same View of explaining Sir *Isaac Newton's* Philosophy. And as the several Propositions were intended to solve some Phænomenon in Nature, I have accordingly applied them to particular Instances taken out of the Third Book, that the Reader may at once see the Excellency and Usefulness of the Author's Method of treating Philosophy. Whoever has a mind to proceed further, and to explain the more difficult and complicated Cases in the System of the World, may go on upon the same Principles, and he will find that if they are rightly applied, there is no Phænomenon, how irregular or minute soever it may seem to be, but may be accounted for by them; but the Propositions here demonstrated are

sufficient to give the Reader a Relish for true Philosophy, and to excite him to pursue the Study of it according to his Abilities and Opportunities. The great Author begins with the most plain and obvious Things, such as every one's own Senses must take notice of, and which every Day's Experience convinces him of the Truth and Certainty of; and he applies them to Cases less obvious or less observed, and shews that they hold equally true and certain in those Cases also. Thus in the common Case of a Traveller, every Man knows that the Number of Miles he travels is in Proportion to the *Pace* he goes, and the *Time* he takes. Now the same Reason must hold likewise in the Case of the Cælestial Bodies, their Travels or Motions in their Orbs must be proportionable to the Pace or Velocity with which they move, and the Time they take up in revolving;
now

now the *Time* being known by Observation, and the *Space* run through, and the *Velocity* bearing a certain Proportion to the Time, they also may be found: and thus the Planets periodical Times may be compared with their Distances, in the same manner as two Travellers may compare their Rates of Travelling with each other. So likewise perpetual Experience teaches us, that all Bodies are heavy, or tend towards the Surface of the Earth, and that their Weights are in Proportion to the Quantities of Matter they contain; now it is very reasonable to think that this is true of all other Bodies, how large soever they be, or at how great a Distance soever they be; for there can no possible Reason be assigned why a two Pound Weight in one Scale should weigh down a single Pound Weight in the other Scale, but the same Reason will hold to prove that the Earth would outweigh the Moon, if they were

were both put in a Balance; which, if it be so, then by the same Rules that the Velocities and Distances, and Times of the Motions of small Bodies vibrating in a *Steelyard* may be computed, the Velocities and Distances, and Times of the Motions of the Planets about each other, and about the common Center of Gravity, may be computed also. These, and such like obvious Truths, are the Principles upon which the incomparable Author builds his whole Philosophy, and they may properly be referred to what is commonly called *Mechanicks*; which though it may be looked upon as a vulgar Term, because it is generally applied to the meanest Trades and Artificers, yet if we duly consider the Nature of the Science, and examine what Opinion the antient Mathematicians had of it, we shall find that it is the Foundation of all true Philosophy, and of the most universal Extent and Use, and so our

The PREFACE

xj

Author esteemed it to be, as appears by the following Description he gives us of it in his Preface. “ The Antients, says he, (as *Pappus* informs us) had a great Opinion of the Usefulness of Mechanicks, in order to their thoroughly searching into the Nature of things; and the Moderns, wholly laying aside substantial Forms and occult Qualities, have attempted to reduce the Phænomena of Nature to the Laws of Mathematicks. The Antients distinguish’d Mechanicks into two sorts, *viz. Rational*, or such as is strictly demonstrable, and *Practical*. In this latter all manual Arts are comprehended, and from them it borrows the Name of *Mechanicks*. But because Artificers do not work very exactly, therefore all Mechanicks are usually so distinguished from Geometry, that when any thing is very exact, it is said to be *geometrically* done; and when it is not so very exact,
“ it

“ it is said to be done *mechanically*.
“ But the want of Accuracy is not
“ the Fault of the Art, but of the
“ Artificers. He whose Works are not
“ very exact, is but an imperfect Me-
“ chanick; and if any one could do
“ a thing with the utmost Exactness,
“ he would be the most perfect Me-
“ chanick that could be. For the
“ Description of streight Lines and
“ Circles, which is the Foundation of
“ all Geometry, belongs to *Mechanicks*.
“ Geometry does not teach us how to
“ describe these Lines, but supposes that
“ we can do it. A Beginner is sup-
“ posed to have learnt how to describe
“ them exactly, before he enters upon
“ Geometry, and then he is taught
“ how Problems may be solved by these
“ Operations. To describe streight
“ Lines and Circles are indeed Pro-
“ blems, but they are not geometrical
“ ones, the Solution of them is bor-
“ row'd from *Mechanicks*, and *Geome-*
“ *try* teaches us the Use of what was
“ before

The P R E F A C E.

xij

“ before solved, and now taken for
“ granted. And it is the Excellency
“ of *Geometry* to be able to perform
“ so many things, from so few Princi-
“ ples borrow'd from another Science.
“ *Geometry* therefore is founded in the
“ Practice of *Mechanicks*, and is no-
“ thing else but that part of *general*
“ *Mechanicks* which proposes and de-
“ monstrates exactly the Rules of Men-
“ suration. But because the manual
“ Arts are principally employed in
“ moving of Bodies, therefore *Geome-*
“ *try* is usually applied to Magnitude,
“ and *Mechanicks* to Motion. Ac-
“ cording to this Distinction, *rational*
“ *Mechanicks* is the Science which pro-
“ poses and demonstrates in the most
“ exact manner the Motions which arise
“ from any Forces whatsoever, and the
“ Forces which are necessary to pro-
“ duce any Motions whatsoever. This
“ part of *Mechanicks* was improved by
“ the Antients in the *five Powers* com-
“ monly called the *mechanical Powers*,
“ and

“ and which are employed in manual
“ Arts ; and they hardly ever confi-
“ der'd Gravity (which is not a manual
“ Power) any otherwise than in order
“ to move Bodies by those Powers.
“ But my Design is not so much to
“ consider Arts as Philosophy, to treat
“ not of manual but of natural Powers,
“ and therefore I shall chiefly examine
“ such things as relate to Gravity, Levi-
“ ty, elastick Force, the Resistance of
“ Fluids, and their attractive or im-
“ pulsive Forces ; and for this Reason
“ it is, that I call these Principles ma-
“ thematical Principles, for the whole
“ Difficulty of Philosophy seems to me
“ to lie in this, that we should find
“ out the Powers or Forces of Nature
“ by the Phænomena, and then from
“ the Forces thus found out, demon-
“ strate other Phænomena ; and with
“ this View it is that I have laid down
“ the general Propositions in the First
“ and Second Books ; and in the Third
“ Book I have produced an Instance of
“ the

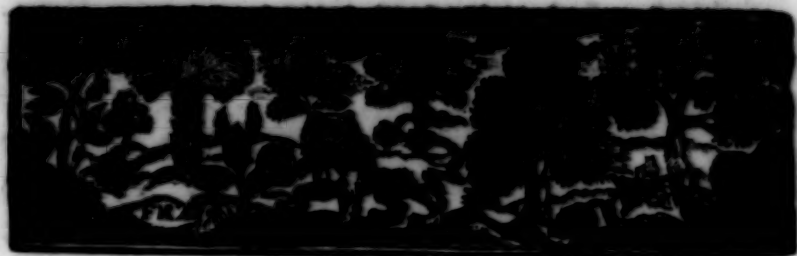
The P R E F A C E.

xv

“ the Truth of what was before de-
“ monstrated, by explaining the System
“ of the World upon those Principles ;
“ for there the Forces of Gravity by
“ which Bodies tend towards the Sun
“ and each of the Planets, are derived
“ from the Phænomena in the Hea-
“ vens, by the Propositions which are
“ mathematically demonstrated in the
“ foregoing Books ; and after that,
“ the Motions of the Planets, Comets,
“ Moon and Sea, are deduced from
“ these Forces by mathematical Proposi-
“ tions also. I wish the remaining Phæ-
“ nomena of Nature could likewise
“ be derived from mechanical Princi-
“ ples, by the same way of arguing.
“ For there are a great many Reasons
“ that induce me to imagine, that
“ they all depend upon certain Forces
“ by which the Particles of Bodies are
“ impelled towards each other, and
“ stick together in regular Figures, or
“ by which they are driven off and re-
“ cede from one another, which arise
“ from

“ from Causes not throughly under-
“ stood ; and these Forces not being
“ known, it is no wonder that Philoso-
“ phers have hitherto in vain attempt-
“ ed to explain the System of Nature.
“ But I hope the Principles here laid
“ down, will give some Light to this Me-
“ thod of Philosophy, or to some other
“ which may be nearer the Truth.”

Now as true Philosophy is not only the most agreeable and pleasant to all those who have any Relish for it ; but is also the most useful and extensive Science ; if the following Sheets will be any Assistance to the Reader in understanding and applying the Principles of it, they will fully answer the End proposed by them.



A N
EXPLICATION

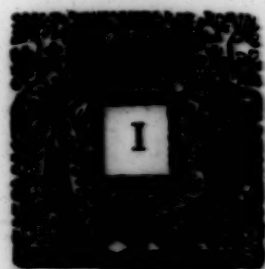
Of some of the Principal SECTIONS

O F

Sir ISAAC NEWTON'S
Principles of Natural Philosophy.

DEFINITION I.

*The Quantity of Matter is to be measured by the
Density and Magnitude together, or by the
Density multiplyed into the Magnitude.*



F Air be made twice as Dense
as it usually is, and the Space
it is contained in be made twice
as large, the Quantity of Air
will be four times as much;
and if the Space be three times
as large, the Quantity will be six times as much.

B

And

D E F I N I T I O N S.

And the same is true in Snow, or any Powders which may be made denser by compressing or dissolving them ; for the like Reason holds for all Bodies which can be any way made denser, by any Causes whatsoever. If M be put for the Quantity of Matter, D for the Density, and B for the Magnitude, it will always be $M = D \times B$. If there be any Medium which freely passes through the Interstices of the Parts of Bodies, I don't now meddle with that at all : But by a *Body* or a *Mass*, I would all along hereafter be understood to mean this Quantity of Matter. And it may always be known by the weight of any Body : For I have found by the most accurate Experiments of Pendulums, that the Quantity of all Matter is in proportion to its weight, as shall be afterwards shown.

D E F I N I T I O N II.

The Quantity of Motion is to be measured by the Velocity and Quantity of Matter together, or by the Velocity multiplied into the Quantity of Matter.

FOR the Motion of the whole Body is the Sum of the Motions of all its Parts, and therefore if the Body be increased to twice the bigness, and the Velocity remains the same, the Motion will be twice as much ; but if the Velocity be increased to twice as much also, then
the

DEFINITIONS.

3

the Motion will be four times as much. And universally, if Q be put for the Quantity of Motion, M for the Body, and U for the Velocity, it will be $Q = M \times U$.

DEFINITION III.

The inherent Force of Matter is that Power of Resistence, whereby every Body, as far as it can, continues in that state in which it is, whether it be a state of Rest, or whether it moves on uniformly in a streight Line.

THE not rightly understanding the nature of this Power in Matter, has been the occasion of many great Mistakes in Natural Philosophy; and this is in a good measure owing to a false Notion generally received concerning Motion, as if Motion were an Action, and as if a Body all the time it is in Motion, were in Action: But if we duly attend to it we shall find, that it is the beginning of Motion only that is an Action, and that to continue it after it is once begun is only Passiveness; and that it is the same thing with Motion as it is with Rest; to begin to be at Rest is as much an Action as to begin to be in Motion, and to continue to be in Motion is exactly the same Passiveness as to continue to be at Rest; they are neither of them Actions at all, but are only the Effects of not being alive; they proceed from the mere inability of Matter to

DEFINITIONS.

alter that state in which it once is, as is evident from hence, that it requires just the same Force to put a Body that is in Motion, into Rest, as it is does to put it out of Rest into Motion; and that as when it is at Rest it can never of it self get into Motion, so when it is once in Motion, it can never of it self get into Rest; nay, it cannot in the least diminish its Motion, nor so much as any way alter the Determination of it: To have a Power of beginning Motion therefore, or which is the same thing, to have a Power of beginning Rest, is to be alive, and to have some degree of Understanding, and both the one and the other are only the same Power differently exercised. If therefore we suppose Matter to be without Life, and that it has only a Power of continuing in Existence merely, without being able to alter any of the Circumstances of its Existence; the necessary Consequence is, that it will equally resist every Alteration or Change of its state whether of Rest or Motion, and that this Resistance will be always proportional to the Body or Quantity of Matter, and differs not at all from its Passiveness, except only in our manner of conceiving it. By this Passiveness of Matter it is, that no Body can, without some Difficulty, be put out of its state either of Rest or Motion; wherefore this inherent Force may very properly be called a Passive Force. This Force is then only exerted in any Body when there is any alteration made in the state

D E F I N I T I O N S.

5

state of it by some other Force impressed upon it ; and the exerting it in this manner may in one respect be stiled Resistance, and in another respect an Impetus or Effort : It may be stiled Resistance as the Body strives, with the impressed Force to continue in that state in which it is ; and it may be stiled an Impetus, as the same Body endeavours to alter the state of any Obstacle in its way, by not yielding to it without difficulty. Resistance is commonly ascribed to Bodies that are at Rest, and Impetus ascribed to those that are in Motion ; but Motion and Rest, as they are commonly apprehended, are only relatively distinguished from each other, for those Bodies are not always at Rest which are generally looked upon to be so.

D E F I N I T I O N IV.

The impressed Force is an Action exercised upon a Body, in order to change its state of Rest or uniform Motion in a streight Line.

THIS Force consists wholly in the Action, and does not at all continue in the Body after the Action is over. For the Body will remain in any new state, by its mere passive Power only. And this impressed Force may proceed from several different Causes, as from a Stroke, from Pressure, or from a centripetal Force.

DEFINITION V.

Centripetal Force is that by which Bodies are from all Parts drawn, impelled, or do any way tend towards some Point as a Center.

GRAVITY by which Bodies tend to the Center of the Earth, is some such sort of Force as this, and so is the Force in a Loadstone or any other magnetick Body ; and so likewise is that Force, whatever it be, by which the Planets are perpetually drawn from their rectilinear Motions, and made to revolve in curve Lines. A Stone whirled about in a Sling, endeavours to fly off from the Hand of him that turns it about, and by so doing it stretches the Sling ; and the quicker it turns about, the more it stretches it ; and as soon as it is let go, it flies off. That Force which is contrary to this, and by which the Sling perpetually draws the Stone towards the Hand, and keeps it in the Orb, I call the centripetal Force, because it is directed towards the Hand as the Center of the Orb. And the same Reason holds for all Bodies that are turned round, they all endeavour to go off from the Centers of their Orbs ; and if it were not for some Force contrary to such Endeavour, by which they are restrained and kept in their Orbs, and which I therefore call Centripetal ; they would actually go off with an uniform Motion in streight Lines. Every projected Body, were it

it not for the Force of Gravity, would not bend towards the Earth, but go on in a streight Line into the Heavens, and that with an uniform Motion, if the Resistence of the Air were removed: It is by its Gravity that it is drawn from its rectilinear Course, and perpetually bent towards the Earth; and it is more or less bent in proportion to its own Gravity and the Velocity of its Motion. The less its Gravity is in proportion to the quantity of Matter, or the greater the Velocity is with which it is projected, the less will it deviate from a streight Line, and the further will it go. If a leaden Bullet be shot out of a Cannon on the Top of a Mountain with a given Velocity, and in an horizontal Direction, and goes in a curve Line to the Distance of two Miles before it falls to the Ground: The same Bullet with twice the Velocity would go twice as far, and with ten times the Velocity would go ten times as far, if the Resistence of the Air were taken away. And by increasing the Velocity, the Distance to which it is projected may be increased as much as you please, and the curvature of the Line described by it may be diminished, so that it will not fall to the Ground till it is arrived at the Distance of ten, or thirty, or ninety Degrees, or till it has even surrounded the whole Earth; or still further, till it will not come to the Ground at all, but go off into the Heavens, and continue to move on for ever. And for the same Reason that a

DEFINITIONS.

projected Body may by the force of Gravity be bent into a Curve, and surround the whole Earth, the Moon may also by the Force of Gravity, if it be heavy, or by any other Force with which it is urged towards the Earth, be perpetually drawn towards the Earth from its rectilinear Course, and bent into its Orb; and without some such Force the Moon could not be kept in that Orb. This Force, if it were too small, would not make the Moon decline enough from its rectilinear Course, and if it were too great, it would make it decline too much, and bring it down out of its Orb towards the Earth, so that it must be of a proper Quantity: And it is the business of Mathematicians to find exactly what Force is sufficient to retain a Body in any given Orb with a given Velocity; and on the other hand, to find the curvilinear Path which a Body may be turned into, when it goes out from any given Place, with a given Velocity, and is bent with a given Force. Now the Quantity of this centripetal Force is of three sorts, absolute, accelerative and motive.

DEFINITION VI.

The Quantity of the absolute, or whole centripetal Force, is greater or less in proportion to the Efficiency of the Cause by which it is propagated from the Center all round, and therefore

DEFINITIONS.

9

fore it is to be measured, or estimated by that Efficiency.

THUS the Magnetick Force is greater or less in one Loadstone than in another, in proportion to the Bulk of the Loadstone, or the Intensity of its Vertue.

DEFINITION VII.

The Quantity of the accelerative centripetal Force, is proportional to, and to be measured by the Velocity which it generates or produces in a given or determinate time.

THUS the Virtue, or Power of a Loadstone is greater at a lesser distance, and less at a greater distance: And the Force of Gravity is greater in Valleys, and less upon the tops of high Mountains, and still less (as will appear afterwards) at greater distances from the Body of the Earth; but it is the same in all Places that are at equal distances, because all Bodies (heavy or light, great or small) when the Resistance of the Air is taken away, fall towards the Ground with equal celerity. Thus in the exhausted Receiver, a small Feather and a piece of solid Gold let fall from the same height, arrive at the Bottom in the same time.

DEFINITION VIII.

The Quantity of the motive centripetal Force is proportional to, and to be measured or estimated

DEFINITIONS.

mated by the Motion which it produceth in a given time.

THUS the larger the Body is, the greater is its Weight ; and the smaller the Body is, the less is its Weight ; and the same Body is heavier near the Earth, than when it is high up in the Heavens. This Quantity is the Tendency or Inclination of the whole Body towards the Center, that is, its Weight ; and may be always known by the Force which is contrary to it, and by which the Body is kept from falling.

THESE Quantities of Forces may for brevity sake be called, motive, accelerative and absolute Forces ; and for distinction, may be referred to the Bodies tending towards the Center, to the Places of the Bodies, and to the Center of the Forces : Thus the motive Force may be considered as in the Body it self, as the tendency of the whole Body towards the Center, composed of the Tendencies of all its Parts : The accelerative Force may be considered as in the Place where the Body is, as some sort of Efficacy diffused from the Center to all the Places round about, in order to move the Bodies which are in them : And the absolute Force may be considered as in the Center, as if that had some Cause in it without which the motive Forces could not be propagated to all the Places round it ; whether this Cause be any central Body (as the Loadstone in the center

D E F I N I T I O N S.

11

center of the magnetick Force, or the Earth in the center of the gravitating Force) or any other Cause whatsoever which is not visible, we do not now consider ; the Notion or Conception is only Mathematical, for we are not now examining into the natural Causes or Places of these Forces.

THE accelerative Force therefore is to the motive Force, as the Velocity is to the Motion ; for the Quantity of Motion proceeds from the Velocity and the Quantity of Matter ; and the motive Force is owing to the accelerative Force, and to the Quantity of the same Matter both. For the motive Force of the whole Body, consists in the Sum of the Actions of the accelerative Force upon every one of the Particles of the Body. Whence it follows, that here upon the Surface of the Earth, where the accelerative Gravity, or the gravitating Force is the same in all Bodies, the motive Gravity, or the weight of any Body, is *as* or *in proportion to* the Body itself : But if we go into the higher Regions where the accelerative Gravity is less, the Weight will be proportionably diminished, and will always be in proportion to the Body and the accelerative Gravity together. Thus in those Places where the accelerative Gravity is but half as much as it is here, the weight of a Body half, or a third Part as big, will be but a fourth or a sixth Part as much.

FURTHER,

D E F I N I T I O N S.

FURTHER, I call Attractions and Impulses in like manner accelerative and motive, in the same Sense of those Words. And I use the Words Attraction, Impulse, or Tendency of any kind towards a Center, indifferently and promiscuously for each other, because I don't consider these Forces Physically but Mathematically. So that the Reader ought to be very careful, when at any time I say either that the Centers attract, or that there are any Forces in the Centers ; not to imagine, that by such Expressions I intend to define any particular sort or manner of Action, or any real Cause or natural Reason in any of those Places, or to ascribe really and physically any Forces to those Centers (which are Mathematical Points only.)

S C H O L I U M.

T H U S far I thought it proper to explain in what Sense, some Words not so commonly known, are to be understood in what follows. Time, Space, Place and Motion are understood very well by every Body. However, we may observe that the generality of Men, have no other Notion of these, than as they respect sensible things ; whence there arise some Prejudices ; in order to take away which, it will be necessary to distinguish them into absolute and relative, true and apparent, mathematical and vulgar.

I. A B S O-

I. **ABSOLUTE**, real and mathematical Time, flows equably or uniformly, of it self, and from the nature of the Thing, without respect to any thing whatsoever that is external to it, and is called Duration: Relative, apparent, or common Time, is some part of this Duration measured by some sensible or external Motion (sometimes exact and sometimes uneven) which is generally used instead of true Time; as an Hour, a Day, a Month, and a Year.

II. **ABSOLUTE** Space continues always alike and immoveable from its own Nature, without respect to any thing whatsoever that is external: Relative Space is any moveable Measure or Dimension of this real Space, and is defined by our Senses, by the situation of it with respect to Bodies, and is taken by the generality for immoveable Space: Of this sort are the Dimensions of any Subterraneous, Aerial or Cœlestial Spaces, which are determined by their situation with respect to the Earth. Absolute and relative Space are the same in kind and in bigness, but they do not always continue to be the same individual Space. For if the Earth (suppose) be moved, the space of our Air, which relatively, and with respect to the Earth, continues always the same, will sometimes be one and sometimes another Part of that absolute Space into which the Air passes; and so the absolute Space of it is perpetually altered.

III. PLACE

DEFINITIONS.

III. **PLACE** is that part of Space which a Body possesses ; and is absolute or relative according as the Space is so ; that part of Space, I say, not the situation of the Body, or the Superficies that surrounds it : For the Places of equal Solids are always equal, but the Superficies are generally unequal, because of the Figures being dissimilar : And the situation can be of no Quantity, properly speaking, nor is it so much the Place as an Affection or Circumstance of the Place. The Motion of the whole Body is the same as the Motion of all its Parts ; that is, the transferring the whole Body out of its Place, is the same thing as the transferring all its Parts out of their Places ; and therefore the Place of the whole Body is the same as the Place of the Sum of all its Parts, and consequently is internal and in the whole Body.

IV. **ABSOLUTE** Motion is the transferring of a Body from one Part of absolute Space to another ; and relative Motion, is the transferring a Body from one Part of relative Space to another. Thus in a Ship under Sail ; the relative Place of a Body, is that Part of the Ship where the Body is, or it is that Part of the whole Cavity of the Ship, which is filled or possessed by the Body, and which therefore is moved along with the Ship ; and relative Rest is the continuance of the Body in the same Part of the Ship, or of its Cavity. But true and real Rest is the continuance of the Body in
I the

the same Part of that immoveable Space in which the Ship it self is moved, together with its Cavity and all the things that are contained in it. If therefore the Earth be really at Rest, a Body which is relatively at Rest in a Ship, will be truly and absolutely moved with the same Velocity that the Ship is moved on the Earth. But if the Earth be moved also, the true and absolute Motion of the Body will arise partly from the true Motion of the Earth in immoveable Space, and partly from the relative Motion of the Ship on the Earth. And if the Body be also moved relatively in the Ship; its true Motion will arise, partly from the true Motion of the Earth in immoveable Space, partly from the relative Motions as well of the Ship on the Earth as of the Body in the Ship; and the relative Motion of the Body on the Earth will arise from these relative Motions. Thus, if that Part of the Earth where the Ship is, be truly moved towards the East with 10010 Degrees of Velocity, and the Ship sails with ten Degrees of Velocity towards the West; and the Pilot walks in the Ship towards the East with one Degree of Velocity; the Pilot will be truly and absolutely moved in immoveable Space with 10001 Degrees of Velocity towards the East, and He will be relatively moved on the Earth with nine Degrees of Velocity towards the West.

IN Astronomy, absolute Time is distinguished from relative Time, by the Equation of
common

common Time. For the natural Days are unequal, though we commonly measure Time by them as if they were equal. This inequality is corrected by Astronomers in order to measure the celestial Motions by Time, which is more true. It is possible that there may be no such thing in Nature as equable Motion by which Time may be accurately measured. All Motions may possibly be accelerated or retarded, but absolute Time flows on equally and uniformly, and cannot possibly be altered. The Duration or Continuance of the Existence of things is the same, whether their Motions be quick or slow, or if they be not moved at all; and therefore this Duration is justly distinguished from those Measures of it by sensible things, and is collected from them by an Astronomick Equation. And the necessity of this Equation, in order to determine the Phænomena, appears by the Experiment of the Pendulum, and by the Eclipses of *Jupiter's* Satellities.

As the Order of the Parts of Time is unalterable, and cannot be disturbed, so also is the Order of the Parts of Space. To move them out of their Places is, (as I may say) to move them out of themselves. For Time and Space are as it were, the Places of their own selves and of every thing else. All Things in the Universe have their Places in Time, with respect to the Order of Succession, and their Places in Space, with respect to the Order of Situation.

Situation. It is essential to Time and Space to be the Places of Things ; and it is absurd for the original Places themselves to be moved ; these therefore are absolute Places, and absolute Motions consist in being transferred from these Places only.

Now because these Parts of Space are invisible, and not to be distinguished from each other by our Senses, therefore we make use of sensible Measures in the room of them : For we define every Place by the Positions and Distances of Things from some Body which we look upon as immoveable ; and so likewise in consequence hereof, we judge of all Motions with regard to the forementioned Places, as we conceive Bodies to be transferred from them. Thus we make use of relative Places and Motions instead of absolute ones ; and this does very well in the common Affairs of Life ; but in Philosophical Matters we ought to think of them abstractly.

ABSOLUTE and relative Motion and Rest may also be distinguished from each other by their Properties and Causes and Effects. It is a Property of Rest, that such Bodies as are really at rest, are also at rest with respect to each other. Wherefore though it be possible that there may be a Body in the Regions of the fixt Stars, or at a much further Distance still, which is absolutely at rest ; yet because we cannot determine from the situation of the Bodies in our own Regions with respect to

C

each

each other, whether any of them always keep in the same Position with regard to that distant Body ; therefore true and real Rest cannot be determined by the situation of these Bodies with respect to each other.

IT is a Property of Motion, that the Parts which keep in a certain or given Position with respect to the whole Body, are moved with the same Motion as the whole Body. For all the Parts of Bodies moving in any Orbs, endeavour to go off from the Axis of Motion ; and the Force, or Impetus of Bodies that move streight forward, arises from the united Force of every one of their Parts. Therefore if Bodies which surround others, are moved, whatever is relatively at rest in the surrounding Bodies, is notwithstanding moved also. Whence it follows that true and absolute Motion cannot be determined by the transferring of a Body from amongst the neighbouring Bodies which appear to be at rest. For these neighbouring external Bodies ought not only to appear to be at rest, but to be really at rest. For if they be not, all Bodies which are included in these surrounding Bodies, will be moved with the true Motions of the surrounding Bodies themselves, at the same time that they are transferred out of the neighbourhood of them ; and if we suppose this transferring of them to be stopped, they will not really be at rest, but only appear to be so. Thus, suppose a Fish swimming in the Sea from West
to

to East, the true Motion of it cannot be judged of by the apparent Velocity in the Water, because both the Fish and the Water move from West to East by the diurnal Rotation of the Earth; and if the apparent Velocity of the Fish be taken away, so that it seems to stand still, it will yet be moved with the true Motion that belongs to the Earth. For the surrounding Bodies are the same with respect to the Bodies contained in them, as the external Part of any Body is to the internal Part of it, or as the Shell is to the Kernel; and if the Shell be moved, the Kernel will be moved also as if it were a part of it, though it be not transferred from the neighbourhood of the Shell.

By a Property very like this it is, that if the Place in which any thing is, be moved, the Thing contained in it, will be moved along with it; and therefore a Body which is moved out of a Place, which Place is moved also, will have the same Motion besides as the Place has. All Motions therefore, which are from Places that are themselves moved also, are only Parts of the whole or absolute Motions; and every whole Motion is compounded of the Motion of the Body out of its original or first Place, and the Motion of that Place out of the Place where it was; and so on, till we come to the Place which is in its own Nature immoveable, as in the foregoing Instance of the Pilot: Whole or absolute Motions there-

DEFINITIONS.

fore cannot be determined but by Places that are immoveable; and for this Reason it is, that I before referred these Motions to Places that are immoveable, and relative Motions I referred to moveable Places. And those Places only are immoveable, the Positions of which are the same with respect to each other from Infinity to Infinity, and must always remain immoveable, and they make that Space which I call immoveable Space.

THE Forces impressed upon Bodies to produce Motion, are the Causes by which true and relative Motion may be distinguished from each other. True Motion can neither be produced nor altered, but by Forces impressed upon the Body in Motion; but relative Motion may be produced or altered without any Forces impressed on such a Body. It is sufficient that the Forces be impressed on other Bodies only to which this stands in some relation, that by their quitting their Places, that Relation in which the relative Rest or Motion consists, may be altered. Further, true Motion is always altered when any Forces are impressed upon a Body in Motion; but it is not necessary that relative Motion should be altered by such Forces; for if equal Forces be so impressed upon other Bodies likewise to which this Body is referred, that the situation with respect to each other continue the same; the Relation in which the relative Motion consists, will continue the same likewise. All relative Motion

Motion therefore may be altered, and the true Motion remain still the same, and it may remain the same when the true Motion is altered ; wherefore true Motion cannot at all consist in such sort of Relations as these.

THE Effects by which absolute and relative Motions may be distinguished from each other, are those Forces by which Bodies moving in Circles recede from the Axis of the circular Motion. For if the Motion in a Circle be merely relative, these Forces will not be seen at all ; but in true and absolute circular Motion, they will be greater or less according to the Quantity of that Motion. If a Bucket be suspended by a long Thread, and turned round till the Thread grows very stiff with being twisted, and be filled with Water so that the Bucket and the Water be both at rest ; and then be on a sudden turned round with some Force the contrary way, and continues for some time in this Motion by the untwisting of the Thread ; we shall see that the Superficies of the Water will at first be plain and smooth, as it was before the Bucket began to move ; but afterwards when the Bucket causes the Water also to begin to turn round visibly, by gradually impressing its Force upon it ; the Water will recede by little and little from the Middle of the Bucket, and ascend at the Sides of it, and will become concave or hollow ; (as I have found by Experience) and the quicker the Motion is, the higher will it ascend, till its Revo-

lutions are made in the same time with those of the Bucket, and then it will be relatively at rest. This ascending of the Water shows that it endeavours to recede from the Axis of Motion, and by this Endeavour we may find out, and measure the true and absolute Motion of the Water as it turns round, and which in this Case is directly contrary to its relative Motion. At the beginning, when the relative Motion of the Water in the Bucket is the greatest of all, this Motion caused no Endeavour to recede from the Axis, the Water did not make towards the Circumference by ascending at the Sides of the Bucket; but it continued smooth, and therefore its true circular Motion was not yet begun. But afterwards, as the relative Motion of the Water decreased, its ascending at the Sides of the Bucket shew that it endeavoured to recede from the Axis; and this Endeavour made it appear plainly, that its true circular Motion increased continually, till it came to be the greatest that it could be, which was when the Water became relatively at rest in the Bucket. Wherefore this Endeavour does not depend upon the Waters being transferred with respect to the Bodies that surround it, and consequently true circular Motion cannot be determined by its being transferred in such a manner. There is but one single truly circular Motion in any revolving Body, and this is correspondent to one single Endeavour, and is the true and adequate

quate Effect of that Endeavour: But there are innumerable relative Motions according to the various Relations, or different Respects of external Things; and like all other Relations, they are wholly unable to produce any true Effects, unless they partake of the true and only Motion. Wherefore in their System of the World who imagine that our Heavens are revolved in an Orb below the Heavens of the fixed Stars, and carry the Planets along with them; all the several Parts of the Heavens, and the Planets which are at rest with respect to the Heavens which are immediately next them, will be moved with a true Motion: For they alter their Positions with respect to each other (which cannot be in Bodies which are truly at rest) and are equally moved along with those Heavens in which they are carried, and as they are Parts of the revolving Wholes, they endeavour to recede from their Axis's of Motion.

RELATIVE Quantities therefore are not really and truly those Quantities by whose Names they are called, but only those sensible Measures of them (whether they be true or false) which are commonly made use of instead of the Quantities themselves which are measured: So that if we would define the Sense of Words by the common use of them, then by the Words Time, Space, Place and Motion, we are properly to understand these sensible Measures; and if we should mean by them, the

DEFINITIONS.

Quantities themselves that are measured, our Discourse would be out of the common way and merely Mathematical. Those Persons therefore who in the Interpretation of Scripture apply these Words to the Quantities measured, do violence to those Sacred Writings. And they equally corrupt Mathematicks and Philosophy, who confound the real Quantities themselves with their Relations and the vulgar Measures of them.

IT is indeed a very difficult thing to find out the true Motion of particular Bodies, and actually to distinguish them from the apparent Motions, because the Parts of that immoveable Space in which Bodies are really moved, are not the Objects of our Senses. However the Case is not entirely to be despaired of. For there may be Arguments fetched partly from the apparent Motions, which are the Differences of the true Motions, and partly from the Forces which are the Causes and Effects of the true Motions. Thus if two Globes, kept at a given distance from each other by a Thread betwixt them, be revolved about the common Center of Gravity; the Endeavour or the Force with which they recede from the Axis of Motion, may be determined by the Tension of the Thread, and from thence the Quantity of the circular Motion may be computed. And if after this any equal Forces be at the same time impressed on the alternate Sides of each of the Globes in order to increase or diminish

minish their circular Motion ; the increase or decrease of the Motion will be found by the Tension of the Thread being increased or diminished ; and hence by repeated Tryals those Parts of the Globes may be found also on which the Forces must be impressed in order to increase the Motion as much as it can be, that is, the hinder Parts of the Globes, or those Parts which follow in the circular Motion. And if the Parts which follow, and the opposite Parts which go before be once found, the Determination of the Motion is also found. After this manner the Quantity and the Determination of such a circular Motion as this, may be found in any immense Vacuum, where there is nothing external and sensible with which the Globes might be compared. But if there were any very distant Bodies placed in that Space, and which kept in a given Position with each other, such as the fixed Stars in the celestial Spaces are ; it could not indeed be determined by the relative transferring of the Globes from some to other of these Bodies, whether the Motion is to be ascribed to the Globes or to the Bodies: But if by observing the Thread it were found that the Tension of it is such as would follow if the Globes were moved ; it might justly be concluded, that the Globes were actually moved and those Bodies at rest ; and then after this the Determination of this Motion might be collected from the transferring of the Globes from some of these Bodies

Bodies to others of them. But how to gather the true Motions from their Causes, Effects and apparent Differences, and on the other Hand how to gather the Causes and Effects from their true or apparent Motions shall be more largely explained afterwards; for the following Treatise was composed for that very Purpose.



AXIOMS,



AXIOMS,

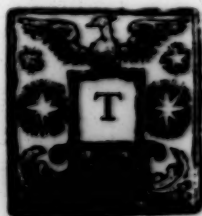
OR

LAWs of MOTION.



LAW I.

Every Body will continue in the State it is in, whether it be at Rest, or whether it moves on uniformly in a streight Line, unless it be made to alter that State by some Force impressed upon it.



HUS all projected Bodies would continue their Motion, if they were not retarded by the Resistance of the Air, and impelled downwards by the Force of Gravity.

And a Top, whose Parts are continually drawn off from their rectilinear Motion by Cohæfion, would

would turn round for ever, if it was not retarded by the Air. And thus the great Bodies of the Planets and Comets continue their Motions both progressive and circular for a very long time, in Spaces where there is very little Resistance.

L A W II.

The Change or Alteration of Motion is proportionable to the impressed motive Force; and is made according to the Direction of the straight Line in which that Force is impressed.

IF any Force produces any Motion, twice that Force will produce twice as much Motion, and three times that Force will produce three times as much Motion; and it is the same thing whether the Forces be impressed all together and by one Stroke, or gradually and one after another. And if the Body was in Motion before, this new Motion (because it is always determined the same way with the Force by which it is generated) will either be added to the Motion of the Body as conspiring with it, or be subtracted from the contrary Motion, or else it will be obliquely added to the oblique Motion, and be compounded with it according to the Determination of them both.

L A W

LAW III.

Re-action or Resistance is always equal and contrary to Action ; or the Actions of two Bodies upon each other are always equal, and directed the contrary way to one another.

IF any Body presses against or draws another Body, it is itself as much pressed against or drawn by that other Body. If a Stone be pressed upon by any one's Finger, his Finger is as much pressed upon by the Stone. If a Horse draws a Stone fastned to a Rope, the Horse may be said to be as much drawn back by the Stone ; for the Rope being equally stretched both ways, will endeavour to loosen itself with the same Force, so as to urge the Horse towards the Stone, just as much as it urges the Stone towards the Horse, and consequently it hinders the Progress of the one exactly as much as it promotes the Progress of the other. If any Body striking against another Body, makes any alteration of the Motion of this other Body by the Force with which it strikes it ; the first Body will in like manner undergo the same alteration in its own Motion, but only it will be the contrary way, by the Force of this second Body ; because the mutual Pressure upon each other is equal. By these Actions of the Bodies upon each other, the alterations that are made in their Motions, not in their Velocities, are equal ; that is, sup-
 posing

posing the Bodies to be no way hindered by any thing else; and the alterations made in their Velocities, which are the contrary ways to each other likewise, are reciprocally proportional to the Bodies, because the Alterations made in the Motions are equal. This Law holds good in Attractions likewise, as shall be proved in the next *Scholium*. Whoever would see this Law of Motion generally demonstrated and applied to particular Cases, both in Elastic and Unelastic Bodies, may find it in the Notes upon *Robault's System of Natural Philosophy*, Chap. XI. Part I. and in Dr. Keil's *Introductio ad veram Physicam*.

C O R O L L A R Y I.

If a Body be acted upon by two Forces, it will describe the Diagonal of a Parallelogram by both the Forces together, in the same Time that it will describe the Sides by those Forces separately.

Fig. 1. IF a Body with the Force M only, impressed upon it in the place A, be carried with an uniform Motion from A to B in a given Time; and with the Force N only, impressed upon it in the same place A, it be carried with an uniform Motion from A to C in the same Time, and the Parallelogram ABDC be compleated; the Body will be carried by both the Forces in the Diagonal AD from A to D in the same time as it would go from A

I

to

to B, or from A to C. For the Force N which is impressed upon the Body according to the direction of the Line AC which is parallel to BD, will make no alteration in the Velocity with which it goes towards this Line BD, and which is produced by the other Force; the Body therefore will come to the Line BD in the same Time, whether the Force N be impressed upon it or no, and therefore at the end of that Time it must be found somewhere in this Line BD; so likewise the Force M which is impressed upon the Body according to the direction of the Line AB, which is parallel to CD, will make no alteration in the Velocity with which it goes towards CD, and which is produced by the other Force, the Body therefore will come to the Line CD in the same time whether the Force M be impressed upon it or no; and therefore at the end of this time, it must be found somewhere in the Line CD; now the same Body cannot be in two different Lines at the same time, unless it be at the intersection of those two Lines, otherwise it would be at two different Places at the same time; wherefore it must necessarily be found at the intersection D. And that it will go with a rectilinear Motion from A to D is evident from Law the first; for it receives its uniform Velocity resulting from both the Impressions in the place A, and it also receives its Determination along the Diagonal AD resulting from the Determinations

nations of the two Impressions in the same place A.

COROLLARY II.

Hence we may see how the direct Force AD is compounded of any oblique Forces AB and BD; and also how any direct Force such as AD may be resolved into any oblique Forces such as AB and BD. And this Composition and Resolution is abundantly confirmed by

Fig. 2.

Mechanicks.

Fig. 3.

THUS, for Instance, suppose O to be the Center of a Wheel, and OM, ON two unequal Spokes going from that Center; let A and P be two Weights hung upon the Ends M and N by the Threads MA and NP; it is required to find the Forces of these Weights in order to move the Wheel. Let the streight Line KOL be drawn through the Center O, and come to the Threads perpendicularly in K and L, and on the Center O with the distance OL, which is the longest of the two Distances OK, OL, let a Circle be described meeting the Thread MA in D, then let OD be drawn, and let AC be drawn parallel to it and perpendicular to DC. Now because the Force of Gravity upon the Weights A and P is impressed according to the Directions of the Threads MA, NP; therefore it is all one whether the Points K, L, D be fastned or not fastned to the Plane of the Wheel, the Weights

will have just the same Forces as if they were suspended on the Points K and L or D and L. Let the whole Force of the Weight A be represented by the Line AD, which suppose to be the Diagonal of a Parallelogram; then (*by the foregoing Corol.*) the Force represented by it, may be resolved into any two Forces whatsoever, according as the Parallelogram is completed: Let these two Forces be represented by the Sides AC, CD; now the Force AC contributes nothing towards moving the Wheel, because it is parallel to DO and draws according to the Direction of it, and is therefore taken off, or supported by the Fulcrum at O. But the other Force DC which draws the Spoke DO perpendicularly, has the same Power as if it drew the Spoke OL perpendicularly, because these two Spokes are of the same length, and the Directions of the Forces are the same; that is, the Force DC is the same as the Weight P, if this Weight P be taken in the same proportion to the Weight A, as the Force DC is to the Force DA, that is, as OK to OD or OL (for the right angled Triangles ADC, DOK are similar, because the Angle DOK is equal to the Angle ADC, each of them being the complement of the Angle KDO to a right Angle, consequently $DC:DA::OK:OD=OL$). If therefore the Weights A and P be reciprocally as the Spokes OK, OL which lie in a streight Line, they will have the same Power and continue *in equilibrio,*

D

librio, which is the known Property of the Scales, Leaver and Windless. And if either of the Weights be greater than in this Proportion, its Power to move the Wheel will be so much the greater.

IF the Weight p be equal to the Weight P , and be partly suspended upon the Thread Np and partly bears upon the oblique Plane pG : then let pH be drawn perpendicular to the Horizon, and NH perpendicular to the Plane pG ; and if the Force of the Weight p tending downwards, or its whole Gravity be represented by the Line pH (a supposed Diagonal of a Parallelogram as before;) this Force may be resolved into the two Forces pN , and HN . Now if there were any other Plane pQ perpendicular to the Thread pN , and cutting the Plane pG in a Line parallel to the Horizon, and the Weight p were not suspended on the Thread pN , but rested wholly upon the two Planes pQ , pG ; it would press upon these two Planes with the perpendicular Forces pN , HN , that is, it would press upon the Plane pQ with the Force pN , and it would press upon the Plane pG with the Force HN . Wherefore if the Plane pQ be taken away, so that the Weight be at liberty to stretch the Thread, it will stretch it with the same Force pN by which the Plane was pressed upon by it before; because the Thread by supporting the Weight now, performs the Part of the Plane that is removed. Whence it follows, that the
Tension

Tension of the oblique Thread pN , will be to the Tension of the other perpendicular Thread PN , as pN to pH . And therefore if the Weight p be to the Weight A , in a proportion which is compounded of the reciprocal proportion of the least Distances of the Threads PN , AM from the Center of the Wheel, and the direct Proportion of pH to pN , that is, if $P : p :: \frac{pH}{KO} : \frac{pN}{OL}$ the Weights will have equal Power to move the Wheel, or will be in *equilibrio*, and will therefore support each other, as any One may try by Experience.

THE Weight p lying upon the two oblique Planes pQ , pG may be compared to a Wedge betwixt the two inward Sides of a Body that is cleaved, and from hence the Forces of the Hammer and the Wedge may be computed : Thus the Force with which the Weight p presses against the Plane pQ , is to the Force by which the same Weight p is urged by its own Gravity, or driven by the Force of a Hammer against the two Planes, as pN is to pH ; and the same Force of the Weight p is to the Force impressed in like manner upon the other Side pG , as pN to NH . So likewise the Force of a Screw may be collected from the like division of the Forces; for a Screw is nothing else but a Wedge assisted by a Leaver. The Use of this Corollary therefore may be extended universally, and the Extensiveness of it plainly shows the truth of it. From what

has been already said may easily be derived the Forces of all such Engines as are made up of Wheels, Cylinders, Pulleys, Leavers, distended Ropes and Weights ascending directly or obliquely, and the rest of the Mechanick Powers; these you may find particularly demonstrated in *Robault's Natural Philosophy*, Chap. XIV. Part I. and in *Dr. Keil's Introductio ad veram Physicam*. And from the same Principles the Forces of the Tendons to move the Bones of Animals may be computed.

COROLLARY III.

The Quantity of Motion which arises from taking the Sum of the Motions which are the same way, and that which arises from taking the Difference of the Motions which are the contrary way to each other, is not at all altered by the Action of Bodies upon one another.

FOR Action and Re-action which is contrary to it, are always equal by Law III, and therefore by Law II, they must produce equal Alterations in the Motions, and towards the contrary Parts, because the Impressions are made the contrary way. Wherefore if the Motions of any Bodies are all the same way, and the one overtake the other and strikes against it, whatever Motion is added to the Body that flies on, must be taken out of the Motion of the Body that follows after, that the
Sum

Sum of the Motions may remain the same as they were before: But if the Bodies meet each other, there must be the same Quantity of Motion taken from each of them, and therefore the difference of the Motions made the contrary ways will remain the same.

THUS suppose the Spherical Body A to be Fig. 4.
 three times as big as the Spherical Body B, and to have two Degrees of Velocity; and let B follow after A in the same streight Line, with ten Degrees of Velocity; then (*by Definition II.*) the Motion of A will be to the Motion of B as six to ten; now suppose the Motion of the one to be six, and that of the other to be ten, then the Sum will be sixteen Degrees of Motion. If therefore upon their meeting the Body A gains three or four or five Degrees of Motion, the Body B will lose just so many Degrees, and consequently the Body A after Reflexion will go on with nine, ten or eleven Degrees of Motion; and the Body B will go on with seven, six or five Degrees; so that the Sum of the Degrees of Motion will always be the same as they were before the Bodies met. If the Body A gains nine or ten or eleven or twelve Degrees of Motion, and so goes on after their meeting, with fifteen, sixteen, seventeen or eighteen Degrees; the Body B, by losing as many Degrees as A gains, will either go on with one Degree, having lost nine, or it will entirely stop, having lost all its progressive Motion of ten Degrees, or it will go back
 D 3 with

with one Degree, having lost all its Motion and one Degree more than all (as I may say;) or it may go back with two Degrees, by having twelve Degrees of Progressive Motion taken from it. And thus the Sums of the Motions conspiring with each other $15 + 1$ or $16 + 0$, and the Differences of the Motions which are contrary to each other $17 - 1$ and $18 - 2$ will always be sixteen Degrees, the same Number as they were before the Bodies met and were reflected. And if the Motions be known which the Bodies will go on with after the Reflexion, the Velocity of each of them may be found, by making it to bear the same Proportion to the Velocity before the Reflexion, as the Motion after the Reflexion bears to the Motion before the Reflexion. Thus in the last Case, where the Motion of the Body A was six Degrees before the Reflexion, and eighteen Degrees after the Reflexion, and the Velocity of it before the Reflexion was two Degrees; the Velocity after the Reflexion will be found to be six Degrees: If we say, as six, the Degrees of Motion before the Reflexion, is to eighteen, the Degrees of Motion after the Reflexion: So is two, the Degrees of Velocity before the Reflexion; to six, the Degrees of Velocity after the Reflexion. All the particular Cases of Reflexion of Bodies overtaking or meeting with each other, are demonstrated at large in the fore-mentioned Book of Dr. Keil's, to which I refer the Reader.

If

IF the Bodies be not Spherical, or if they move in different streight Lines, and strike against each other obliquely; and it be required to find their Motion after Reflexion; we must first know the Situation of the Plane which the Bodies touch in the Point where they meet: Then (*by Corol. 2.*) the Motion of each Body is to be distinguished into two Motions, the one perpendicular to this Plane, and the other parallel to it: The parallel Motions are to be continued the same after Reflexion as before, because the Bodies act upon each other in the Direction of the Line perpendicular to this Plane; and the Alterations in the perpendicular Motions towards the contrary Parts, are so to be reckoned, that the Sum of the Motions the same way, and the Difference of the Motions the contrary ways, may continue the same as they were before the Bodies struck against each other. From such sort of Reflexions as these, the circular Motions of Bodies about their own Centers, do commonly arise likewise; but I shall not have occasion to consider these Cases in the following Treatise, and it would be too tedious to demonstrate every Particular that relates to these Matters.

A LEMMA to the following COROLLARY.

If two streight Lines AC, BD be terminated at the given Points A, B, and be to each other in a given Proportion; and if the streight Line CD by which the indeterminate Points

D 4

C, D

CD are connected, be divided in a given Proportion in K: I say the Point K will always be in a streight Line given in Position.

Fig. 5. FOR let the streight Lines AC, BD meet in E, and in BE let BG be taken in the same Proportion to AE as BD is to AC, and let FD be always equal to the given Line EG; then by the Construction of the Figure, EC will be to GD, that is, to EF (which is equal to GD) as AC to BD; therefore EC will be to EF in a given Proportion and consequently (the Angle CEF remaining always the same) the Triangle EFC will be given *in Specie*, or all the Triangles EFC, Efc will be similar. Let CF be so divided in L, that CL may be to CF in the same Proportion that CK is to CD, and let the Line cf be divided in l; then because LF is to lf in that given Proportion, and the Angle LEF always the same, the Triangles EFL, Efl will be similar or given *in Specie*; and therefore (by the Elements) the Points L, l, will always be in the streight Line ELl given in Position, or whose Situation always remains the same. Let LK be joined, and the Triangles CLK, CFD will be similar; (because the Sides CL, CK; CF, CD about the same Angle are Proportional); for the same Reason, if lk be joined, the Triangles clk, cfd will be similar also; and because FD is given, that is, is equal to fd, and the Proportion between LK and FD is

is given or is the same as that between lk and fd , therefore LK is given, or is equal to lk . Let EH be taken equal to LK , or lk , and $ELKH$, E/kH , will always be a Parallelogram: The Point K therefore will be in the side of this Parallelogram HK which is given in Position.

COROL. Because the Figure $EFLC$ is given in *Specie*, that is, all the Angles are given, and the Proportion of the Sides to each other are given also, therefore the three streight Lines EF , EL and EC , that is, Gd , HK and EC are in given Proportions to each other, that is, they are to each other as Gd , Hk and Ec .

COROLLARY IV.

The common Center of Gravity between two or more Bodies, does not alter its State of Rest or Motion by the Action of those Bodies upon each other; and therefore the common Center of Gravity of all Bodies that act upon each other, is either at Rest or moves on uniformly in a streight Line (unless it be hindered from so doing, or be acted upon by any thing external.)

FOR if two Points, as A, B move along Fig. 6. the streight Lines AC, BD with an uniform Motion, and CD their distance from each other be divided in a given Proportion in K ; this Point K thus dividing them, will (by the fore-

foregoing Lemma) either be at rest or move on uniformly in a streight Line, if the Motion of the Points A, B be in the same Plane; and the same Demonstration will hold good if the Motions be not in the same Plane, for the Figures they describe will still continue similar, and the correspondent Sides proportional to each other. Therefore if there be ever so many Bodies A, B, D, F moving on uniformly in streight Lines, C the common Center of Gravity of any two of them, as A, B will either continue at Rest, or move on uniformly in a streight Line; because the Line A B which connects the Centers of these Bodies, moving on uniformly in a streight Line, is divided in C in a given Proportion by this common Center. So likewise E the common Center of Gravity of these two Bodies and any third Body D, will either be at Rest or move on uniformly in a streight Line, because C D the distance of the common Center of Gravity C of the two Bodies A, B from the third Body D is divided in a given Proportion in E. In like manner G the common Center of Gravity of the three Bodies A, B, D and any fourth Body F, will be either at Rest or move on uniformly in a streight Line a b; because E F the distance between the common Center of Gravity of the three Bodies A, B, D and the Center of the fourth Body F is divided in a given Proportion in G, and so on for ever. Wherefore in a System of Bodies which do not any way act upon each other externally,

ternally, and are entirely free from any foreign Impressions, and consequently do all of them move on in distinct streight Lines, the common Center of Gravity of them all will either continue at Rest, or will move on uniformly in a streight Line.

FURTHER, In a System of two Bodies acting upon each other, as A, B; because the Distances of the Centers of each of them from the common Center of Gravity, is reciprocally as the Bodies; that is, C A is to C B, as the Body B to the Body A; therefore the relative Motions of these Bodies, either when they approach towards the Center C or remove further from it, will be equal; (for these Motions are as the Bodies multiplied into the Velocities with which they move) and therefore the Center C is neither moved forward or backward, nor undergoes any Alteration in its State either of Rest or Motion, from the Actions of the Bodies A and B upon each other. So likewise in a System of several Bodies, such as A, B, D, F, because the common Center of Gravity C of any two of them, as A, B, acting upon each other mutually, does no way alter its State by such mutual Action; and H the common Center of Gravity of the other two Bodies D, F, which that mutual Action of A and B is not between, cannot undergo any Alteration from it; and C H the distance of these two Centers is divided by G the common Center of Gravity of all the Bodies, into Parts reciprocally proportional

Fig. 7.

portional to the whole Sums of the Bodies whose Centers they are, that is, GH is to GC as the Sum of the Bodies F, D , to the Sum of the Bodies B, A , and therefore these two Centers H and C , maintaining their State either of Motion or Rest, G the common Center of them all, will maintain its State likewise: whence it is manifest that G the common Center of Gravity of all the Bodies, will never alter its State either of Rest or Motion, from the Action of two Bodies upon each other. Now in such a System, all the Actions of the Bodies upon each other, are either Actions between two Bodies, or are compounded of Actions between two Bodies, and therefore they can make no Alteration in the State of the common Center of them all, as to its Motion or Rest. So that, seeing this Center is either at Rest, or moves on uniformly in some streight Line, when the Bodies do not act upon each other, this same Center will continue to be at Rest or always to go on uniformly in a streight Line, notwithstanding the Actions of the Bodies upon each other, if they be not put out of this State by some external Forces impress'd upon the System. Therefore as to a Continuance in a State of Motion or Rest, the Law is the same in a System of a great many Bodies as in a single Body; for the progressive Motion of a single Body, or of a System of Bodies, is to be reckon'd by the Motion of the Center of Gravity.

COROL.

COROLLARY V.

If any Number of Bodies be included in a given Space, their Motions amongst each other will be the same, whether that Space be at rest, or moves on uniformly in a streight Line, without any circular Motion.

For the difference of the Motions of the included Bodies, if they move the same way, or the Sum of their Motions if they move contrary ways, is the same at the beginning, whether the Space in which they are included moves or be at rest (*by the Hypothesis*;) but the Blows or Forces with which these Bodies strike each other arise from these Sums or Differences; and therefore (*by Law II.*) the Effects of their meeting each other will be the same in both Cases; consequently their Motions amongst each other in the one Case will be equal to their Motions amongst each other in the other Case. The same thing may be proved by a very clear Experiment; all the Motions which are in a Ship are exactly the same, whether the Ship be at rest, or moves on uniformly in a streight Line.

COROLLARY VI.

If any Number of Bodies be moved amongst each other in any manner whatsoever, and at the same time be urged on in parallel Lines
by

AXIOMS, or LAWS of MOTION.

by equal accelerative Forces, or if they be all of them equally accelerated in the same Direction; they will all of them continue to be moved in the same manner amongst or with respect to each other, as if they had not been urged by those Forces.

FOR these Forces by acting equally (in proportion to the Quantities of the Bodies to be moved) and in the Direction of Lines parallel to each other, will move all the Bodies equally (with respect to their Velocity) by the IId Law, and therefore they can make no Alteration in the Positions or Motions of the Bodies with respect to each other.

SCHOLIUM.

HITHERTO I have laid down such Principles as have been received by Mathematicians, and confirmed by abundance of Experiments. By the two first Laws and the two first Corollaries, *Galilæus* found that the Descent of heavy Bodies is in a duplicate Proportion of the Time, or is as the Square of the Time, and that the Motion of projected Bodies is in a Parabola, except only that these Motions are a little retarded by the Resistence of the Air. In a falling Body the uniform Gravity impresses equal Forces upon it, by acting equally in every equal Particle of Time, and also produces equal Velocities: and therefore in the whole Time of

I

its

its falling, the Gravity impresses the whole Forces, and produces the whole Velocity proportional to the Time. And the Spaces described by the Fall in different Times, are as the Velocities and the Times together; that is, in a duplicate Proportion of the Times. Let the whole Time be divided into very small equal Particles, then if the Line AB be divided into an equal Number of equal Parts Aa , ad , &c. the whole Line AB will represent the whole Time of the Body's falling; and any part of the Line AB as AC will represent any proportional Part of the Time of its falling; In like manner, if the whole Line BD represent the Sum of all the equal Velocities generated by the uniform Force of Gravity, and this Line be divided into as many equal correspondent Parts Bf , fg as the Line AB , then will these small Parts represent the equal Velocities produced in the Particles of Time Aa , ad answering to them; and because the Space through which a Body moves is as the Time multiplied by the Velocity, therefore if the whole Space through which the Body falls, be divided into an equal Number of Parts as the Line AB or BD is; each of these Parts may be represented by the correspondent Particle of Time multiplied by the correspondent Velocity, that is, the Space which the Body falls through in the Time Aa is as the Rectangle $Aa \times ab$, and the Space it falls through in the Time Cc is as the Rectangle $Cc \times CE$ or $CEec$, and conse-

Fig. 8.

consequently the whole Space through which the Body falls will be represented by the Triangle ABD, and any proportional Space as AC by any proportional Part of this Triangle, viz. by the Triangle ACE: Then because these Triangles are similar, AC will be to CE, as AB to BD, that is, the Velocities will be as the Times; consequently the Spaces which are as the Velocities multiplied into the Times, will be as the Times multiplied into the Times, that is, as the Squares of the Times. In the same manner may it be demonstrated likewise, that when any Body is thrown upwards, the uniform or continually acting Power of Gravity impresses Forces upon it, and decreases its Velocity, or takes away Velocities from it, in proportion to the Times; and that the Times of its ascending to its greatest Heights, are as these Velocities thus taken away by Gravity; and also that the greatest Heights it can ascend to, are as the Velocities and Times together, or as the Squares of the Velocities. So likewise, if a Body be projected according to the Direction of any straight Line, the Motion arising from such Projection is compounded with the Motion arising from Gravity. Thus if a Body A by its projectile Motion alone would describe the straight Line AB, and by the Motion of its Fall alone, would describe the Height AC, both in the same time; let the Parallelogram ABDC be compleated, and the Body A by a Motion compounded of these two, will be found in the

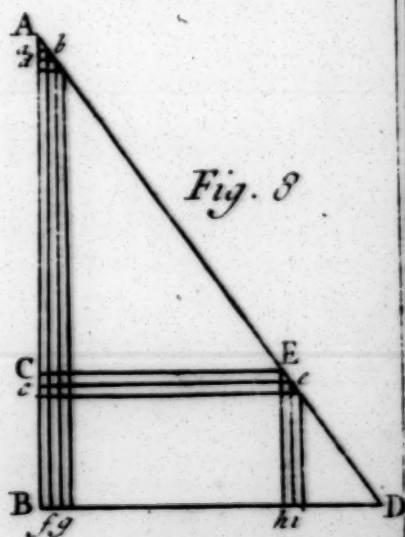
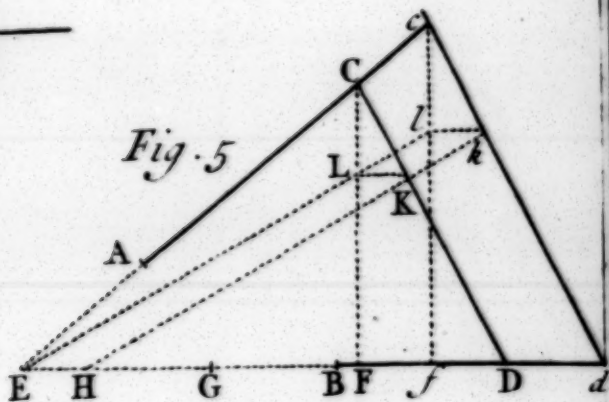
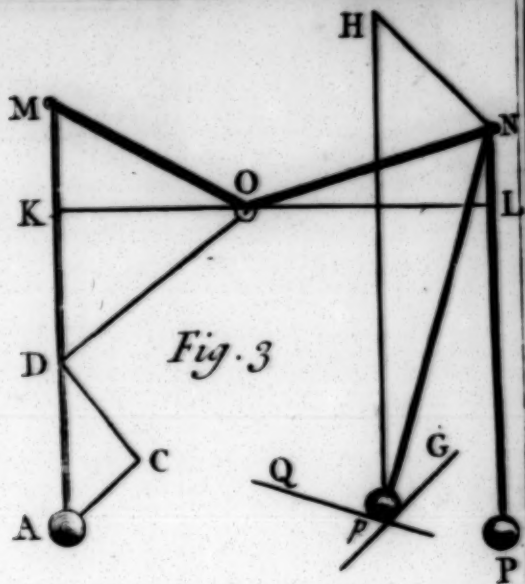
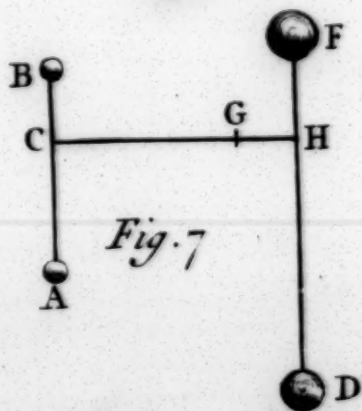
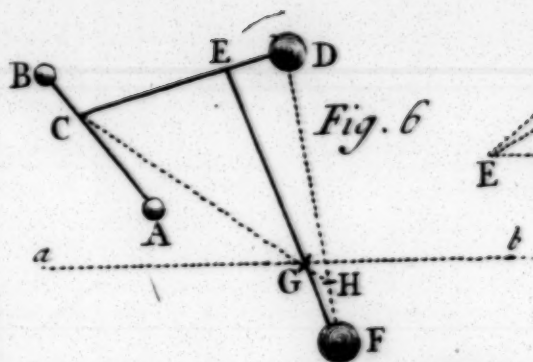
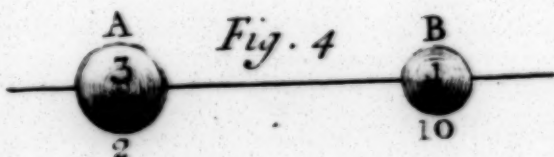
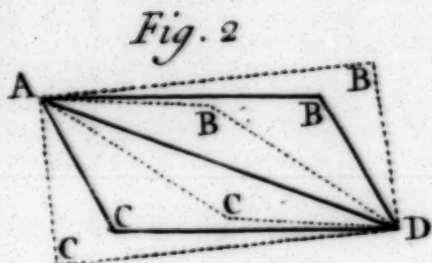
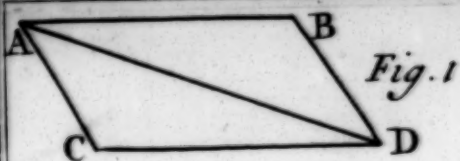
I

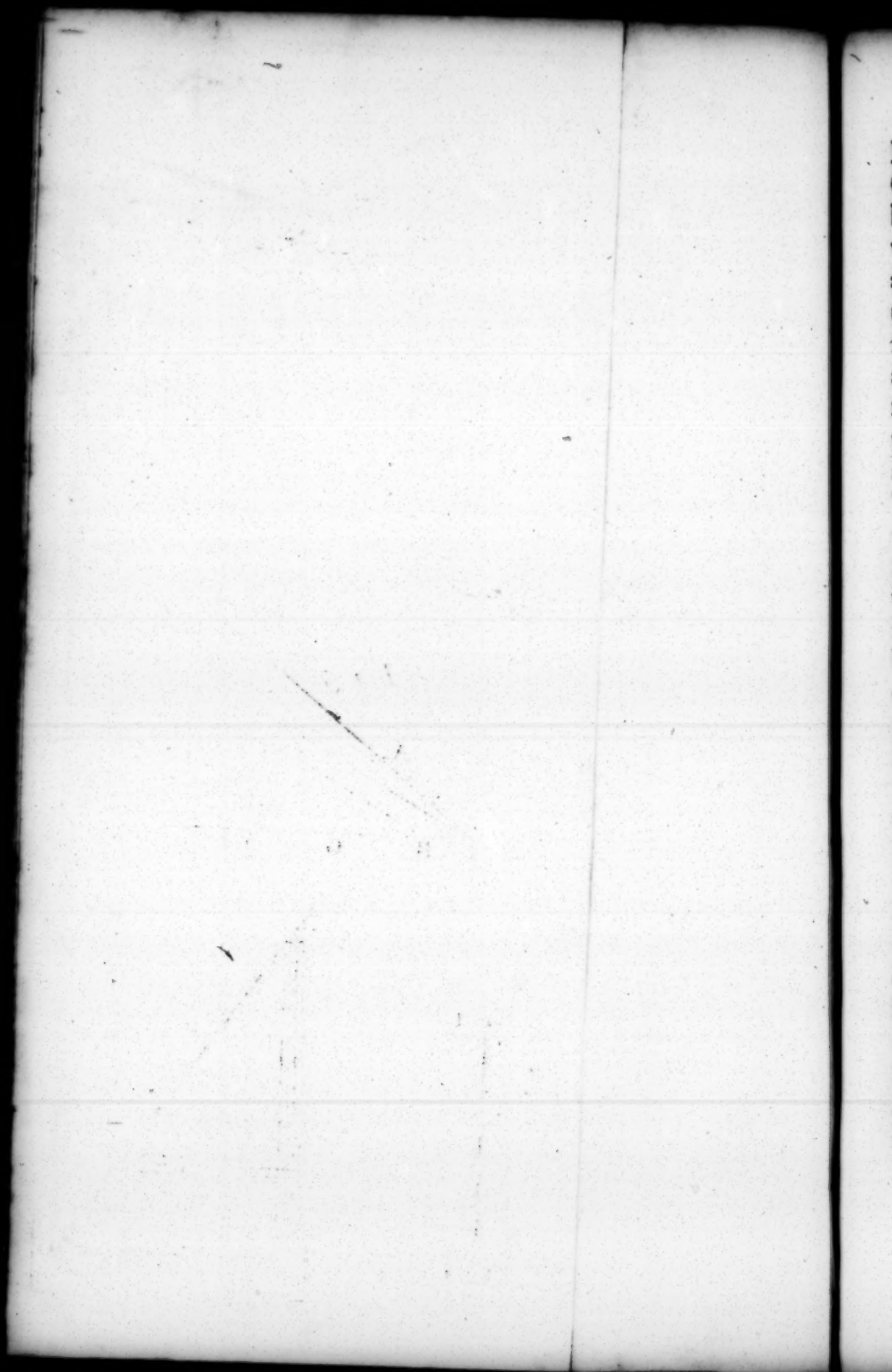
Place

Fig. 9.



the
gle
by
the
les
D,
es;
o-
he
as
n-
en
or
es
or
n
-
-
-
-
-





Place D at the end of this Time, and the Curve Line ADE described by this Body will be a Parabola, which the streight Line AB will touch in A, and the Ordinate BD will be as AB square. For we have just now shown, that the Space AC which the Body A describes in falling by the uniform Force of Gravity, is as the Square of the Time, and this Motion not being at all hinder'd by the projectile Motion from A to B, but only its Determination altered; it is plain, that at the end of that Time, the Body must by the Action of Gravity have described a Space equal to AC; wherefore if the projectile Force would have carried the Body to B, and BD be taken equal to AC, it must by the compound Motion of both Forces, be found in D; (*by Law III. Cor. 1.*) but AB or CD being described by a given Velocity, is proportional to the Time in which it is described, or to the Time of the Body's falling from A to C; therefore AB square is proportional to BD, which is the Property of the Parabola. (*See Part I. Sect. 2. Prop. 19. of Milns's Conick Sections.*) And upon the same Laws and their Corollaries depend all those Propositions which are demonstrated concerning the Times of the Oscillations of Pendulums, confirmed by the daily Experience of Clocks. And from these same Corollaries and the third Law, Sir Christopher Wren, and Dr. Wallis, and Mr. Huygens, the great Geometricians of the last Age, did each of them find out the Laws

E observed

observed by hard Bodies in striking against each other and being reflected; and they communicated them to the *Royal Society* almost at the same time, and agreed exactly with one another (so far as regarded these Laws.) Dr. *Wallis* was the first that shewed his Discoveries to them, and afterwards Sir *Christopher Wren* and Mr. *Huygens* did the same. Sir *Christopher Wren* proved the Truth of his by the Experiment of Pendulums before the *Royal Society*, which was afterwards explained by the famous Mr. *Mariot* in a whole Book. But in order that this Experiment may agree exactly with the Theories, we must take into the Account, both the Resistence of the Air, and the elastick Forces of the Bodies which strike against each other. Let two spherical Bodies A, B hang on the Centers C, D by the parallel and equal Threads A C, B D; on these Centers, and with the Distances A C, B D, let the Semicircles E A F, G B H be described and bisected by the Radius's A C, D B. Let the Body A be lifted up to the Point R in the Arch E A F, and (removing the Body B out of the way) let fall from thence, and after one whole Oscillation or Vibration, let it return to V; then will R V be the Retardation from the Resistence of the Air: Let S T be the fourth part of R V, and so placed in the middle between R and V, that R S may be equal to T V, and that R S may be to S T as three to two. Then will S T represent the Retardation made in the Bodies descending

Fig. 10.

scending from R to A very nearly. For the Pendulum in moving backwards and forwards, will describe an Arch equal to four times the Arch RA, wanting the small Space RV; wherefore ST a fourth part of this Space will very nearly represent the Resistance made in the Bodies Descent from R to A; I say, very nearly; for in order to represent it exactly, we must consider, that the longer the Space is through which the Body moves, the more Resistance will it meet with from the Air; wherefore as the Motion of a Pendulum grows slower and slower, and the Arches described by it less and less, it must meet with less and less Resistance every Oscillation. Let the Arch RV therefore be divided into four equal Parts Ra, ab, bc, cV; then because RA is the longest of the four parts of the whole Oscillation, Ra will be somewhat too small to represent the Retardation made in the Descent from R to A; on the other hand VA being the shortest of the four parts of the whole Oscillation, cV which is equal to Ra, will be somewhat too big to represent the Retardation made in the Ascent from A to V. In like manner bc will be somewhat too big to represent the Retardation made in falling from b to A. Whence it follows, that if the Point S be taken in the middle betwixt a and b, and also the Point T in the middle between b and c, ST will be a fourth part of the whole Retardation, and will

represent the Retardation made in the Descent of the Body from S to A in the exactest manner that can be: Let the Body B be put into its place again, and let the Body A be let fall from the Point S and its Velocity in A , the Place where it is reflected, will be the same without any sensible Error, as if it had fallen from T in a Vacuum. Now let this Velocity be represented by the Chord of the Arch TA . For it is a Proposition well known to Geometricians, (*and demonstrated in Robault, Part II. Chap. 28. Prop. V. of the Notes*) that the Velocity of a Pendulum in the lowest Point is as the Chord of the Arch which it has described in falling. After Reflexion, let the Body A come to the place s , and the Body B to the place k . Let the Body B be removed out of the way again, and let the place v be found, from which if the Body A be let fall, and after one whole Oscillation it returns to r , rt may be a fourth part of rv , and situate in the middle of it, so that rf and tv may be equal; and let the Velocity which the Body A had immediately after its Reflexion in the place A be represented by the Chord of the Arch tA ; for t will be the true and correct place to which the Body A would ascend, if the Resistance of the Air were taken away: By the same Method; the place k is to be corrected, to which the Body B will ascend, and the place l to be found to which the same Body would ascend in a Vacuum,

cuum. By this Method we may make all sorts of Experiments in the same manner as if we were really in a Vacuum. Lastly, the Body A must be multiplied (as I may call it) into the Chord of the Arch T A which represents its Velocity, and this will give us the Quantity of its Motion in the place A just when it is going to be reflected; and after that it must be multiplied into the Chord of the Arch t A, and that will give us its Motion in the place A immediately after it is reflected. In like manner the Body B must be multiplied into the Chord of the Arch B I, in order to find the Quantity of its Motion immediately after it is reflected. And by the same Method, when two Bodies are let fall from different places, the Motions of each of them are to be found both before and after Reflexion, and then these Motions may be compared with each other, and the Effects arising from Reflexion may then be computed. By trying in this manner with Pendulums of ten Foot long, and that in unequal as well as equal Bodies, and by making these Bodies meet after they were let fall from the greatest Distances, as eight, twelve or sixteen Foot; I always found, within a less Errour than three Inches length, where the Bodies met each other directly, that the Alterations of the Motions made in the Bodies towards the contrary parts, were equal, and consequently that Action and Reaction are always equal. Thus if the

Body A fell upon the Body B at rest, with nine parts of Motion; and losing seven of those parts of Motion, went on after Reflexion with two parts; the Body B flew back with those seven parts. If the Bodies met each other, A with twelve parts of Motion and B with six, and A came back with two parts, then B came back with eight parts, fourteen parts being subtracted from each of them: Thus if twelve parts be taken from A, there will remain nothing; then let two parts more be taken away, and that will make two parts of Motion the contrary way: in the same manner, if from the six parts of Motion in the Body B, there be taken fourteen, there will remain eight parts the contrary way. If the Bodies moved both the same way, A the swiftest with fourteen parts, and B the slowest with five parts, and after Reflexion A went on with five parts, B would go on with fourteen parts, nine being transferred from A to B. And so in all other Cases. The quantity of Motion which arise from the Sum of the Motions made the same way, and the difference of the Motions made the contrary ways, was never altered by the Bodies meeting and striking against each other. For I ascribed the Errour of one or two Inches in the measure, to the difficulty of doing all the things exactly. For it is very difficult to let fall both the Pendulums at the same time, so that the Bodies should strike against each other in the lowest place A B, and it is also difficult
to

to mark the places *s, k* to which the Bodies ascended after they met. And in the pendulous Bodies themselves, the unequal Density of their Parts, and the irregular Texture arising from other Causes, must make some Errours.

FURTHER, lest it should be objected that the Rule which this Experiment was invented to prove, supposes Bodies to be entirely hard, or at least to be perfectly elastick, such as are not to be found in natural Compositions; I add, that the Experiments just now described will succeed in soft Bodies as well as in hard, for they do not at all depend upon the Condition of Hardness. For if the Rule be tried in Bodies not perfectly hard, nothing more is required but that the Reflexion should be diminished in a certain Proportion to the quantity of the elastick Force. By the Theory of Sir *Christopher Wren* and Mr. *Huygens*, Bodies which are absolutely hard go back from each other with the same Velocity with which they meet. This may with greater Certainty be affirmed of Bodies which are perfectly elastick. In Bodies which are not perfectly elastick, the Velocity with which they return must be diminished as the elastick Force diminishes; because this Force (if the parts of the Bodies are not injured by striking against each other, or be not any way extended as if they were struck with a Hammer) is certain and determinate (so far as I can perceive) and makes the Bodies return from each other with a relative Velocity, which

is in a given or constant Proportion to the relative Velocity with which they meet. I tried this in the following manner with Balls of Wooll wound up very hard and tied about very streight. In the first place I found the elastick Force, by letting fall the Pendulums and measuring the Reflexion; and after that I determined by this Force what the Reflexions would be in other Cases when they met, and the Experiments succeeded. The Balls always returned from each other with a relative Velocity, which was to the relative Velocity with which they met, as five to nine or thereabouts. Balls made of Steel returned very nearly with the same Velocity: others made of Cork returned with a little less Velocity; and in some made of Glas the Velocities were as fifteen to sixteen very nearly. And in this manner the third Law which respects Bodies striking against each other, and Reflexions, is proved by the Theory, which plainly agrees with Experiments.

Fig. 11. IN Attractions the same thing may be shown in very few Words thus. Suppose the Bodies A and B to attract each other mutually, and any Obstacle C to be placed between them, so as to hinder them from meeting and striking against each other. If either of the Bodies, as A be attracted with a greater Force towards the other Body B, than this other Body B is to the first Body A; then the Obstacle C will be more pressed upon by the Body A than by the Body B, and consequently cannot remain *in equilibrio*:
the

the stronger Pressure must prevail over the weaker, and cause the System or Composition of the two Bodies and the Obstacle, *viz.* A, C, B to move in a streight Line towards that side which B is of, and to go on with an accelerated Motion for ever in empty Space; which is absurd and contrary to the first Law of Motion. For by this Law, the System ought to continue in its State of Rest, or to move on uniformly in a streight Line *a b*, and therefore the Bodies must press upon the Obstacle equally on each side, and consequently must be equally drawn towards each other. I tried this in a Loadstone and Iron; if they be put separately into proper Vessels which just touch each other, they will float together in standing Water as if they were but one Body; neither of them will drive forward the other, but they will each of them support their mutual Attractions towards one another each way, and at last when they are come to an Equilibrium they will be at rest. In the same manner is the Gravitation betwixt the Earth and any part of it, mutual: for suppose the Earth F I to be divided Fig. 12. by any Plane E G into two parts E G F, E G I, the Weight of these two parts upon each other will be equal: for if the greater part E G I be divided into two other parts E G K H and H K I, by another Plane H K parallel to the first Plane E G, so that the part H K I may be equal to the part E F G that was first cut off: then is it manifest that the middle part E G K H will not
 by

by its own Weight tend towards either of the extreme Parts, but will (as I may say) be suspended and remain at rest, as being *in equilibrio* between them both. But the extreme part H K I will bear upon the middle part with its whole Weight, and urge it towards the other extreme part E G F, and therefore the Force with which E G I, the Sum of the parts H K I and E G K H, tends towards the third part, is equal to the Weight of the part H K I, that is, equal to the Weight of the third part E G F which is equal to H K I. Whence it follows, that the Weight of the two parts E G I, E G F upon each other is equal. And if their Weights were not equal, the whole Earth floating in free *Æther*, would yield to the greatest Weight, and by fleeing from it, would go on for ever, in the streight Line *a b*.

As when Bodies strike against each other, and when they are reflected, their Powers are equal, if their Velocities be as their inherent Forces; (*See the III^d Definition*;) so in all mechanical Instruments, the Powers acting upon them are equal, and will mutually support each other by their contrary Efforts, if their Velocities, measured according to the Determination of their Forces, be reciprocally as their Forces. Let B F, F C be two equal or unequal Arms of a Balance vibrating on the Axis or Point of Suspension F, and let the Powers (Weights suppose) to move them be applied to

Fig. 13. the Extremities B and C; then the Direction of Gravity

Gravity being according to the Lines AB , CD , if these Lines be taken in the same Proportion to each other that AF bears to FD , that is, if the Triangle AFB be made similar to the Triangle FCD ; then will CD represent the Velocity downwards, and BA the Velocity upwards, with which the Extremities B , C of the Balance are moved in passing out of the Situation BFC into the Situation AFD ; for these Lines CD , BA are the Spaces passed through in a given Time: And therefore if a Weight be applied to C , which bears the same Proportion to a Weight applied to B , that the Line AB bears to the Line CD , or that BF bears to FC (because the Triangles ABF , FCD are similar;) these Weights will support each other, because the quantity of Motion upwards and downwards is equal (by *Definition II.*) and will consequently destroy each other: that is, the Powers of Weights ascending and descending directly upwards and downwards are equal, if the Weights be reciprocally as the Distances of the Points on which they are suspended, from the Axis of the Balance. But if such Weights ascend or descend obliquely on oblique Planes, or any other Obstacles which hinder their direct Ascent or Descent, they will *then* support each other, when they are in proportion to one another reciprocally as their perpendicular Ascents and Descents, because the Determination of Gravity is downwards. For let ABC be a right-angled Triangle in which

Fig. 14.

which CB represents the Plane of the Horizon, AC a Plane perpendicular to it, and AB an inclined Plane; let D and E be two Weights connected by the Thread DAE moving freely by means of a Pulley fixed at A. Then if EG be taken equal to FD, it is manifest, that in the same time that the Body D descends from F to D by the Force of Gravity, the Body E will ascend on the inclined Plane AB from G to E; let the Line FE be drawn through the Points F, E and continued towards H, and from the Point G let a Perpendicular be let fall intersecting the Line FH in H. Then will the Line FD represent the Velocity with which the Body D descends perpendicularly by the whole Force of Gravity, and the Line HG will represent the accelerative Force by which the Body E descends along the inclined Plane AB; the Determination of which is perpendicular also, because it arises solely from Gravitation; wherefore if the Body E bears the same proportion to the Body D that the Line FD or EG bears to the Line HG, these Bodies will continue *in æquilibrio*, or will support each other (by *Definition II.*) because their Velocities multiplied into the Quantities of Matter are equal. So likewise in a Pulley, or a Windless, consisting

Fig. 15, of several Pulleys; if the Force of the Hand
16. pulling the Rope directly downwards, be in the same Proportion to the Weight ascending either directly or obliquely, as the Velocity of the perpendicular Ascent of the Weight, to the

the perpendicular Velocity of the Hand pulling the Rope, then will the Force of the Hand exactly support the Weight; Thus let P be a Pulley or Windle, W a Weight to be lifted up by the Hand A, and BC an inclined Plane on which the Weight slides; if in the same time that the Hand descends from D to H, the Weight ascends from c to a; the Lines DE, *ab*, will represent the perpendicular Velocities of the Descent of the one, and the Ascent of the other; and therefore (by *Def. II.*) if the Force with which the Hand pulls the Line be to the Weight W as *ab* is to DE, this Force will just support the Weight. For the same Reason, in Clocks and such like Machines, consisting of a great Number of Wheels applied to each other; if two Forces, the one to promote and the other to hinder the Motion of the Wheels, be to each other reciprocally as the Velocity of the parts of the Wheels on which they are impressed, they will support each other. Thus if the Circle described by the Extremity of the Hand of a Clock be twenty four Inches, and if whilst this Hand describes such a Circle, the Weight which keeps the Clock going sinks an Inch: A Force applied to the Extremity of the Hand, equal to a twenty-fourth part of the Force that would lift the Weight from the Ground, will stop the Clock. Again, in the Case of a Screw; the Force of the Screw to press upon any Body, is to the Force of the Hand which turns about the

1

Leaver,

- Leaver, as the circular Velocity of the Leaver in that part which the Hand is applied to, is to the progressive Velocity of the Screw towards the Body pressed upon. For let DEBFH represent a Screw turned about by the Leaver HB; and whilst the Hand at H moves with a circular Motion from H to C, let the Screw rise from G to F, and by pressing upon the Weight A raise it up to the same height; then by what has been said, FG will be to HC, as the Force made use of by the Hand, to the Force of the Screw upon the Weight A, or which is the same thing, as the Weight A upon the Screw.
- Fig. 18. In like manner if EFG represent a Wedge, and ABCD a Block of Wood to be rived by it, and let the Perpendiculars Gb, Gi be let fall from G to the Lines eg, fg. It is evident, that in the same time that the Wedge sinks from G to g by the Force of the Hammer, the parts of the Wood separate from each other by the Spaces bG, Gi, and therefore Gg is to bG + Gi, as the Velocity with which the Wedge enters the Wood, to the Velocity with which the parts of the Wood separate from each other; and consequently (by what has been already said) the Forces with which the Wedge urges the two sides of the Wood to be cleaved, are to the Force of the Hammer that strikes the Wedge, as the Entrance of the Wedge into the Wood, according to the Direction of the Force impressed on it by the Hammer, is to the Velocity with which the parts of the Wood yield

F H



b

x

B

c

aver
s to
ards
re-
B;
ru-
ife
ht
at
ce
e
e
y.
e
y
e

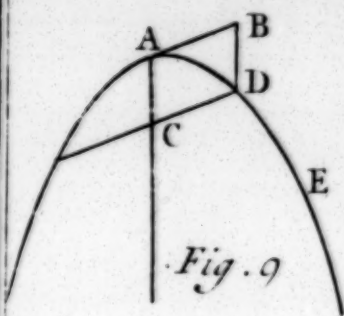


Fig. 9

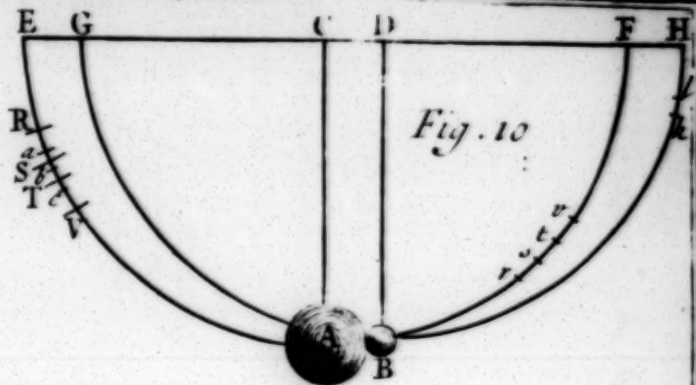


Fig. 10

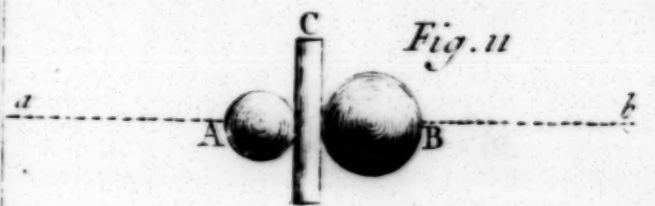


Fig. 11

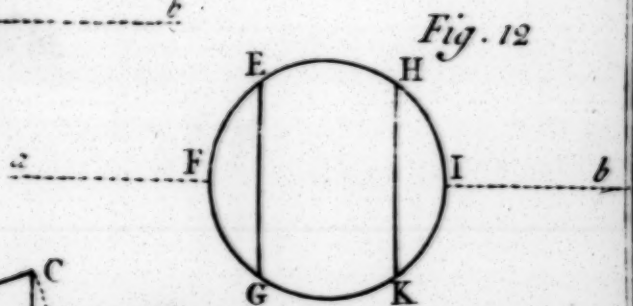


Fig. 12

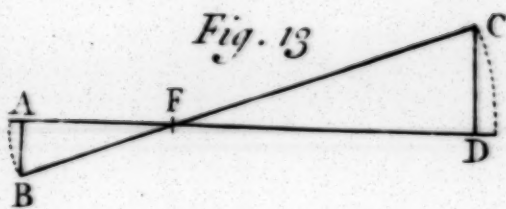


Fig. 13

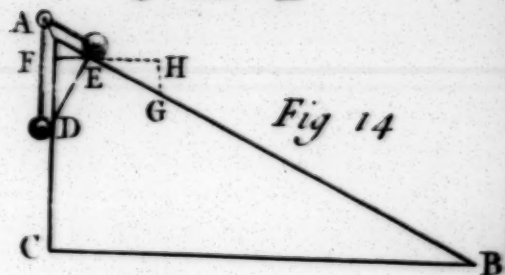


Fig. 14

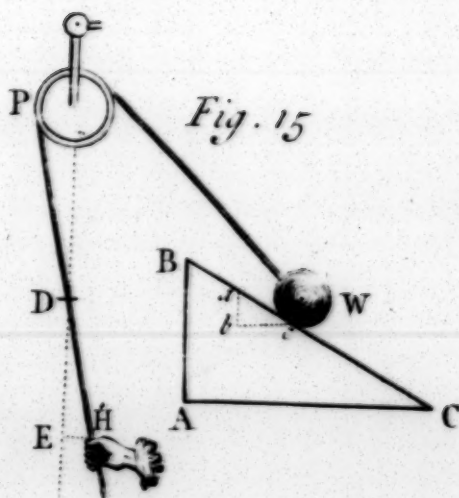


Fig. 15

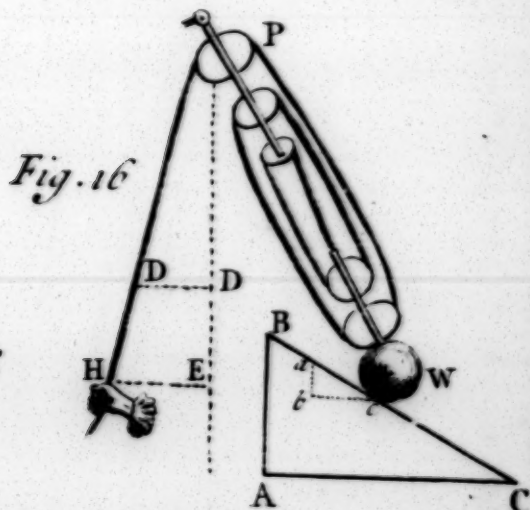


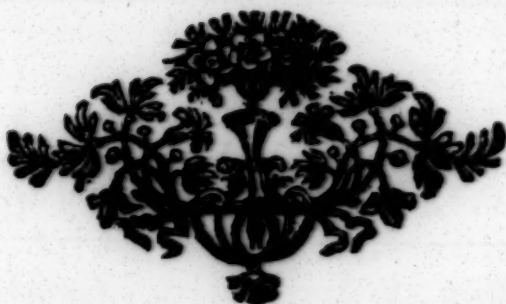
Fig. 16

Y
P
t
I
t
v
I
t
t

yield to the Wedge, in the Direction of Lines perpendicular to the sides of the Wedge. And the like Reasoning holds good in all sorts of Engines whatsoever.

THE Usefulness and Excellency of such Instruments and Engines consist wholly in this, that they increase the Force by diminishing the Velocity, and *the contrary*: whereby this Problem; *To move a given Weight with a given Force*, may be solved by any proper sort of Instrument. For if the Instruments be so formed, that the Velocities of the Person working the Instrument, and of the Body that resists it, be reciprocally as the Forces; his Strength will support the Resistance, and He will overcome it, if the difference of the Velocity be greater. And without doubt, if the difference of the Velocities be so great, as still to overcome all the Resistance which commonly arises from the Friction of contiguous Bodies rubbing against each other, and that which arises likewise from the Cohæſion of solid Bodies which are to be separated from one another, and that of the Weights to be raised beside; all this Resistance being overcome, that Force which still remains, will produce an Acceleration of the Motion proportionable to it self, some of it amongst the several parts of the Engine, and the rest in the resisting Body. But it is not my Design to treat of Mechanicks in this Place. I only intended to shew how universally extensive and certain the third Law of Motion is. For if

the Action of the Person working these Instruments be estimated by the Force he employs, and the Velocity joined together or multiplied by each other; and on the other hand, if the Reaction of the resisting Body be estimated by the Velocities of all its Parts, and by the resisting Forces arising from their Attrition, Cohæ-sion, Weight and Acceleration; Action and Reaction will always be found equal to each other, in every Instrument that can be made use of. And as the Action is continued by and conveyed through the Instrument, and at last impressed upon some resisting Body; the last Determination of it, will always be contrary to the Determination of the Reaction.





B O O K I.

Concerning the MOTION *of* BODIES.



S E C T I O N I.

*Of the Method of first and last Proportions,
by means of which the following Propositions
are demonstrated.*



BEFORE we come to demonstrate these Proportions, it will be necessary to explain the Nature of them, that we may have a clear Notion of them, and be able to see the Force of the Demonstrations; because the whole Reasoning in the following Book is built upon these Proportions, and the Truth of the several Propositions depends entirely upon them. As the Design of the great Author was to explain the principal Phænomena of Nature, so the Prin-
F principles

ciples which he builds upon, are taken from the obvious Reason and Course of things, which every one who attends to, cannot but see and experience; and from a few plain and clear Principles He draws the most surprising Conclusions, and explains the most difficult Appearances in Nature. Thus whoever considers the Nature of *Time*, will see that it is not capable of being made to move faster or slower, but that the very small Parts of it, or the *Moments* of it, are all equal: Wherefore *Time*, and the *Moments* of it, may be represented by any thing that moves uniformly, and whose very small Parts correspond to the *Moments* which the Time is made up of. Thus if a streight Line, or a part of the Circumference of a Circle, be supposed to be generated by the uniform Motion of a Point, (whose Magnitude is not consider'd, but only the Velocity with which it moves) such a streight Line, or such a part of the Circumference of a Circle, may represent any given Time; and as this Time may be divided into any Number of Moments, so also may the streight Line, or the part of the Circumference of a Circle, be supposed to be divided into exactly the same Number of very small Parts; consequently all the Reasonings about such streight Lines and parts of Circumferences of Circles, and all the Demonstrations concerning them, and all the Proportions that arise from comparing them with each other, will equally hold good, and be
be

be exactly the same in *Time* and its *Moments* compared with each other. So likewise, if (as in *Definition II.*) *Motion* be consider'd, as compounded of the quantity of Matter and its Celerity, that is, if the Motion of any Body be as the Quantity of its Matter multiplied into the Celerity with which it moves; then may one side of a Parallelogram be made to represent the Quantity of Matter, and the other side be made to represent the Velocity, and consequently (*by the Elements*) the Parallelogram it self will represent the whole Motion; the Parallelogram divided by one side, will represent the Velocity; and divided by the other side will represent the Quantity of Matter: And in this manner, the whole Motions, the Velocities and the Quantities of Matter in Bodies may, by means of Parallelograms, be compared with each other and their Proportions estimated. For the same Reason, the Space through which any Body moves, the Time it takes up in moving through that Space, and the Velocity with which it moves, may also be represented by a Parallelogram and its sides, because this Space is always proportionable to the time multiplied by the Velocity. By such plain and easy Steps as these, may we go on to things more difficult and compounded, but which are grounded upon the same obvious Principles. We observe by continual Experiences, that all sorts of Figures are described by the Motion of Bodies; the most natural Way therefore to find out the Pro-

perties of those Figures, or to compare their Sides or Area's with each other, and to determine the Magnitudes of them, must be to consider them as generated in this manner. Thus if we imagine a *physical Point*, or the smallest Particle of Matter that we can conceive, to move out of its place in any Direction or with any Velocity, it will by such Motion generate a Line streight or curved; and if we suppose this Line to move in the same manner out of its place in any new Direction, or one Extremity of it being fixed, the other to turn round, it will by its Motion generate a Superficies either plain or curved; and if we suppose this Superficies either to move out of its place likewise in any other Direction, or to turn about any of its Sides, it will generate a Solid whose Superficies will be either plain or curved, or compounded of both: This is a much more natural way of considering Quantities, and leads us more directly to find out the Properties of Curves, Superficies and Solids, and to compare them together, than either the Method of the Antients by the Sections of Solids; as in the Cone for instance, a Section of which parallel to the Base, produces a Circle; a Section through the Axis produces a Triangle; a Section inclined to the Axis produces an Ellipsis, till it becomes so much inclined as to be parallel to the side of the Cone, and then the Section becomes a Parabola, or if it be still more inclined, it then becomes an Hyperbola: or than the Method

thod of the Moderns, who consider Lines as composed of small Parts placed by the sides of each other, and Superficies as made up of such Lines, and Solids as made up of such Superficies: For we see Lines described by the Motions of Bodies every day, and Angles by the Rotation of Lines, and the like: And if we consider further, that such Quantities generated in this manner in a given Time, become greater or less in Proportion as the Velocity with which they are generated is greater or less, this will lead us to find out the Magnitudes of the Quantities by the Velocities of the Motions with which they are generated; of which Method there will be many Instances given in the following Treatise. But to proceed: As Time moves on uniformly and continually, the Moments of it are infinitely small, and as the Space through which any Body moves may be divided into an infinite Number of parts, and as there are infinite Degrees of Velocity, these must all be taken into Consideration, before we can have a true and compleat Notion of the generation of Quantities. The Instances of Gravitation and the Motion of any Planet about the Sun, will make this Matter plain. If a Body descends towards the Earth by the mere Force of Gravity, it appears by undeniable Experiments, that this Force not only acts continually upon the Body, but is also continually increased the nearer it approaches the Earth; so that all the while it is falling, there is not

only a Repetition of the Force impressed upon it every Moment of Time, and in every Point of Space, but also an Increase of that Force in a certain proportion to the distance of the falling Body from the Center of the Earth: Wherefore if we would compute the Quantity of this Force in any given Body, it is necessary to compare the very small parts of Space described in the several Moments of Time, with each other, in order to find out the whole Space described and the whole Velocity acquired. So likewise in the Motion of any Planet about the Sun, or any Body revolving about a Center in a Circle or Ellipsis. It is evident from the first Law of Motion, that the most natural Motion of such a Body is in a streight Line, with the Velocity and in the Direction first given to it. And consequently if it be made to move in a Curve Line, this must be effected by a continual Impression upon it, whereby its Determination is continually altered; equally if it moves in a Circle, and unequally if it moves in any other Curve; because the Curvature of a Circle is every where equal, but the Curvature of other Curves is unequal; wherefore in this Case likewise, it is necessary to consider how much the Body deviates from a streight Line in every Moment of Time and in every Point of Space, during the whole Course of its Revolution, in order to compute its Velocity, and the length it has described in any given Time. It was these, and such like Considerations, that put our great

Author

Author upon finding out finite Quantities which are proportional to these Moments of Time, and to the Velocities generated in them, and to the Spaces moved through in them; and the Proportions of these momentary Velocities and Spaces are what we are to understand by *first or last Proportions*, or which is the same thing, the Proportions which the Velocities with which Quantities (that are generated by local Motion,) begin to be generated, bear to each other; or, if the Quantities decrease, the Proportions which the Velocities with which they decrease, bear to each other, at the Instant that such Quantities vanish or become equal to nothing. If the Motions by which any two Quantities (as two Lines suppose) are generated, be uniform, that is, if the Velocities of the generating Points be equable, the Lines generated in the same time, whatever the lengths of them be, will be to each other as the Velocities with which they are generated. But it is not necessary that Velocities with which Lines are generated, should be always uniform, they may increase or decrease in any Proportion whatsoever, and it is the Design of the following *Lemma's* to determine the Proportion of these Velocities, the Moment the Quantities begin to be generated, and the Moment they end or are entirely compleated: The Quantities are here supposed to be finite, to be generated or produced by certain Velocities in given or determinate Times. Before they begin to be ge-

nerated they are nothing; after they are begun, we do not here enquire what Proportions they bear to each other; but it is their Proportions in the intermediate Point of Time, that we here consider; that is, the Proportions or Velocities with which they begin to be generated, and these are called their *first Proportions*. In like manner, when finite Quantities decrease and at last come to nothing, by local Motion in given Times, there must be certain single Moments in which they are actually vanishing, before which they are not at the Conclusion, and after which they are concluded; the Proportions which the Velocities of such Motion bear to each other in these Moments, are called the *last Proportions*. Now if we attend to it, we shall see, that in this manner of generating Quantities, the Velocities and the Quantities generated by them, do all along accompany each other, and are inseparably connected (though we can in our own Minds consider them apart) and consequently the Quantities throughout the whole Times in which they are generated, are perpetually increased, and the Increments are every where *as* or in proportion to the Velocities in the several Moments of these Times; therefore the *Increments* may be put instead of the Velocities to which they are proportional; and the Proportions of the first of these Increments in the first Moments of the Times when the Quantities are increasing, are also the *first Proportions*; and the Proportions of

of the last of these Increments in the last Moments of the Times, when the Quantities are decreasing, are also *the last Proportions*. I have been thus particular in explaining the nature of these Proportions, because the whole Reasoning of the following Treatise is founded upon them; and the *Lemma's* which we now come to demonstrate, will be so many Illustrations of what has been said.

L E M M A I.

Quantities, and also the Proportions which Quantities bear to each other, if they continually approach nearer and nearer to an Equality in any given or determinate Time; and before that Time is ended, are so nearly equal, that the difference between them is less than any assignable Quantity whatsoever: These Quantities and these Proportions, will in the end be exactly equal.

IF you deny it; then suppose them at the end of this Time to be unequal, and let the last difference of them be called D: Then they cannot approach nearer to Equality than by the given Difference D, which is contrary to the Hypothesis. It is supposed in this *Lemma*, that the Time in which the Quantities are decreasing is finite, and that when this Time expires, they vanish entirely and become equal to nothing: for it is possible to suppose the Time infinite, and that the Quantities are continually approach-

approaching nearer and nearer to each other, in such a certain Proportion, as never to become equal. Thus for Instance: Let CBA be an Hyperbola, DdA its Asymptote, and DB , db Lines drawn from any Points Bb parallel to the other Asymptote; it is evident (by Prop. XXVIII. Part I. Sect. 2. of *Milns's Conick Sections*) that ZD multiplied by DB is equal to Zd multiplied by db ; consequently ZD is to Zd as db is to DB ; and this Proportion holding every where, it follows, that the Lines DB and db , though they continually approach nearer and nearer to each other, yet can never become equal, nor consequently can any part of the Asymptote dA ever become equal to or coincide with its correspondent part of the Curve

Fig. 20. bA . But in the Circle IAG , if the Tangent AD be drawn, this Line touches the Circle in the point A , by the Definition; and if from any point in the Circle as B , the Line BD be let fall perpendicular to the Tangent in D , and this Line be supposed to move parallel to it self, till it coincides with the point A ; it is manifest, that when it is coincident with the point A , it is entirely vanished, as is likewise the Tangent DA and the Arch BA which are terminated by it. Let bd be its Situation the Moment before it vanishes, then by this *Lemma* the difference betwixt the Tangent dA and the Arch bA is less than any assignable Quantity, or bears no finite Proportion to DA or BA ; after bd is vanished, dA and bA are both become equal to nothing;

nothing; therefore in the intermediate Moment, or the Moment they are vanishing, they must be equal, because they cannot arrive at nothing till they have passed through Equality. Fig. 21.

COROL. Hence it follows, that if there be two Lines AB, CB so near to Equality that their difference AC is less than any assignable Quantity, or is so small that no Multiple of it whatsoever will make it equal to either of the Lines AB, CB, that is, if AC bears no finite Proportion to these Lines, then in all our Arguments about such Lines, they may be supposed equal, and put indifferently for each other.

LEMMA II.

If in any Figure AacE contained within the straight Lines Aa, AE and the Curve Line acE, there be inscribed any number of Parallelograms Ab, Bc, Cd, &c. composed of equal Bases AB, BC, CD, &c. and the sides Bb, Cc, Dd, &c. parallel to the side of the Figure Aa; and if the Parallelograms aKbl, bLcm, cMdn, &c. And after this, the breadth of these Parallelograms be diminished, and their Number increased in infinitum: I say that the last Proportions which the inscribed Figured AKbLcMdD, the circumscribed Figure AalbmcndoE and the curve-lined Figure AabcdE, bear to each other, are the Proportions of Equality, or these Figures will all at last become equal..

FOR

FOR the difference betwixt the inscribed and circumscribed Figures, is composed of all the Parallelograms Kl , Lm , Mn , Do , and these Parallelograms are equal to a Rectangle whose Base is Kb , the Base of one of the Parallelograms, and its Altitude Aa , the Sum of the Altitude of all the Parallelograms (because the Bases of them all are equal) that is, to the Rectangle $ABla$. But this Rectangle is less than any given Quantity, because its breadth AB is infinitely diminished. Therefore (by *Lemma I.*) the inscribed and circumscribed Figures, and much more the intermediate curve-lined Figure, will at last become all equable. W. W. D.

LEMMA III.

The same last Proportions, are also Proportions of Equality, though the Breadths of the Parallelograms AB , CB , CD , &c. be unequal, if they be all diminished infinitely, or do all of them become infinitely small.

Fig. 22. FOR let AF be equal to the Breadth of the biggest of them, and let the Parallelogram $FAaf$ be compleated; this Parallelogram will be bigger than the Parallelogram Aa/B , which is the difference betwixt the inscribed and circumscribed Figures; but if AF the breadth of this, becomes infinitely small, the Rectangle will become less than any given Rectangle. W. W. D.

Corol.

Corol. 1. Hence it follows, that the Sum of all the Parallelograms at the last when they vanish, will every way coincide with the curvilinear Figure, or become equal to it.

Corol. 2. And the rectilinear Figure, which is contained within the streight Lines aK , AB , and all the Chords of the vanishing Arches or infinitely small Arches ab , bc , cd , &c. will much more coincide at last with the curvilinear Figure.

Corol. 3. In like manner will the circumscribed right-lined Figure contained within the same streight Lines aK , AB , and all the Tangents belonging to the same Arches ab , bc , cd , &c. coincide at last with the same curvilinear Figure.

Corol. 4. Wherefore these Figures in their last or vanishing State, are not (with respect to their Perimeters acE) rectilinear, but the curve-lined Limits or Bounds of right-lined Figures.

LEMMA IV.

In the two Figures $AacE$, $PprT$, let there be ^{Fig. 23,} inscribed (as before) two Series of Parallelo-^{24.} grams, and let the Number of them be the same in both Figures; and when the breadths of them become infinitely small, let the last Proportions of the several corresponding Parallelograms in each Figure, be equal: I say, that the two Figures $AacE$, $PprT$, are to each other in the same Proportion as those infinitely small Parallelograms.

FOR

FOR since the Parallelogram $AebB$ bears the same proportion to the Parallelogram $PmqQ$, that the Parallelogram $BfcC$ bears to the Parallelogram $QnrR$; (by *Composition*) the Parallelogram $AebB$ will bear the same proportion to the Parallelogram $PmqQ$, that the Sum of the Parallelograms $AebB$, $BfcC$ bear to the Sum of the Parallelograms $PmqQ$, $QnrR$, and by the same Argument one whole Figure will bear that same Proportion to the other whole Figure; for (by *Lemma III.*) the first Figure is composed of the Sum of all its single Parallelograms, and the second Figure is composed of the Sum of all its single Parallelograms. W. W. D.

Corol. Hence it follows, that if two Quantities of any kind whatsoever be divided any way into the same number of Parts, and those Parts, after the number of them is increased, and the bigness of them is diminished infinitely, bear a given Proportion to each other, the first part of the one Quantity to the first part of the other, the second part of the one Quantity to the second part of the other, and so for all the rest in their Order; the whole Quantities will be to each other in that same given Proportion. For if in the Figures belonging to this *Lemma*, the Parallelograms be taken in the same Proportion to each other as the parts of the Quantities, the Sum of the parts will always be as the Sum of the Parallelograms; and therefore when the number of the parts of the Quantities and the

the Parallelograms of the Figures are increased, and their Magnitude diminished infinitely; these Sums will be to each other in the last Proportion of a single Parallelogram in one Figure to its correspondent single Parallelogram in the other Figure, that is, (*by the Hypothesis*) in the last Proportion of a single part of one Quantity to a single correspondent part of the other Quantity.

L E M M A V.

In similar Figures, all the homologous or corresponding Sides, whether they be curvilinear or rectilinear, are proportional to each other, and the Area's are in proportion to the Squares of such Sides.

FOR all similar Figures are composed either of similar Triangles or of similar Sectors of Circles, or of other Curves, and such Triangles and Sectors have (*by the Elements*) their correspondent Sides proportional, and their Areas are in a duplicate proportion to such Sides. W. W. D.

L E M M A VI.

*If any Arch ACB given in Position, be sub- Fig. 25.
tended by the Chord AB, and a streight Line
AD produced both ways, touches this Arch in
any point A in the middle of a continual unin-
terrupted Curvature; and after that the
points A and B approach towards each other
and*

FOR since the Parallelogram $AebB$ bears the same proportion to the Parallelogram $PmqQ$, that the Parallelogram $BfcC$ bears to the Parallelogram $QnrR$; (by *Composition*) the Parallelogram $AebB$ will bear the same proportion to the Parallelogram $PmqQ$, that the Sum of the Parallelograms $AebB$, $BfcC$ bear to the Sum of the Parallelograms $PmqQ$, $QnrR$, and by the same Argument one whole Figure will bear that same Proportion to the other whole Figure; for (by *Lemma III.*) the first Figure is composed of the Sum of all its single Parallelograms, and the second Figure is composed of the Sum of all its single Parallelograms. W. W. D.

Corol. Hence it follows, that if two Quantities of any kind whatsoever be divided any way into the same number of Parts, and those Parts, after the number of them is increased, and the bigness of them is diminished infinitely, bear a given Proportion to each other, the first part of the one Quantity to the first part of the other, the second part of the one Quantity to the second part of the other, and so for all the rest in their Order; the whole Quantities will be to each other in that same given Proportion. For if in the Figures belonging to this *Lemma*, the Parallelograms be taken in the same Proportion to each other as the parts of the Quantities, the Sum of the parts will always be as the Sum of the Parallelograms; and therefore when the number of the parts of the Quantities and the

the Parallelograms of the Figures are increased, and their Magnitude diminished infinitely; these Sums will be to each other in the last Proportion of a single Parallelogram in one Figure to its correspondent single Parallelogram in the other Figure, that is, (*by the Hypothesis*) in the last Proportion of a single part of one Quantity to a single correspondent part of the other Quantity.

L E M M A V.

In similar Figures, all the homologous or corresponding Sides, whether they be curvilinear or rectilinear, are proportional to each other, and the Area's are in proportion to the Squares of such Sides.

FOR all similar Figures are composed either of similar Triangles or of similar Sectors of Circles, or of other Curves, and such Triangles and Sectors have (*by the Elements*) their correspondent Sides proportional, and their Areas are in a duplicate proportion to such Sides. W. W. D.

L E M M A VI.

If any Arch ACB given in Position, be sub- Fig. 25.
tended by the Chord AB, and a streight Line
AD produced both ways, touches this Arch in
any point A in the middle of a continual unin-
terrupted Curvature; and after that the
points A and B approach towards each other
and

and coincide: I say, that the Angle BAD contained between the Chord and the Tangent, will be diminished infinitely, and at last will entirely vanish.

Fig. 25. FOR if the Angle BAD contained betwixt the Chord and the Tangent does not vanish, the Arch ACB will make an Angle with the Tangent AD equal to a right-lined Angle, and then the Curvature at the point A will not be uninterrupted, which is contrary to the Hypothesis.

L E M M A VII.

The same things being supposed; I say, that the last Proportion that the Arch, the Chord and the Tangent bear to each other, is that of Equality, or when they vanish they are all equal.

Fig. 26. FOR whilst the point B is approaching to the point A, let us imagine that all that time the Lines AB and AD are produced to the points *b* and *d* at a good distance, and let *bd* be drawn parallel to the Secant BD. And let the Arch *Ac b* be always similar to the Arch ACB. Then if the points A and B coincide, the Angle *dAb* will vanish, (*by the foregoing Lemma;*) and therefore the streight Lines *Ab*, *Ad*, which are always finite, and the intermediate Arch *Ac b* will coincide, and consequently will be equal; whence it follows likewise, that the streight Lines AB, AD and the intermediate Arch ACB (which are, by the Hypothesis, similar

A

—

B

E

D

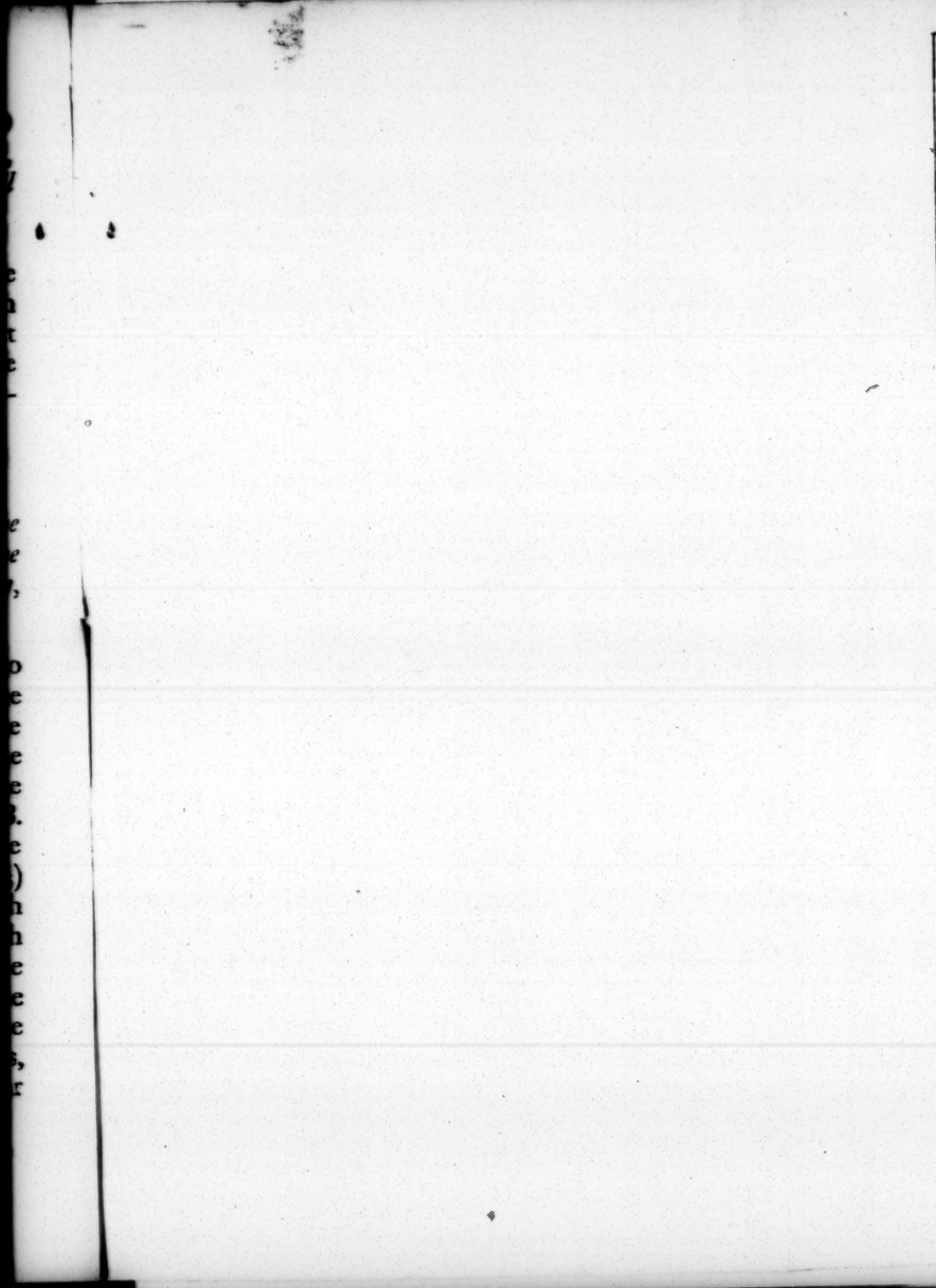


Fig. 17.

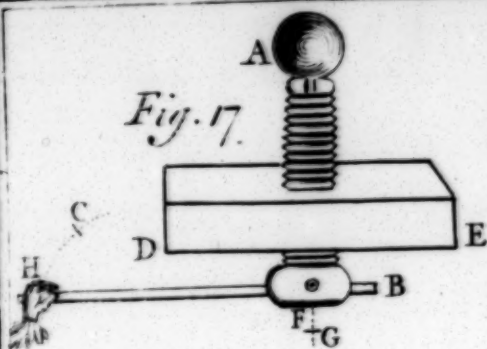


Fig. 18

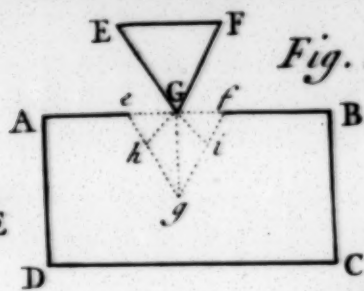


Fig. 19

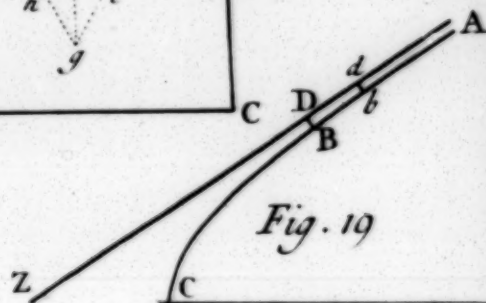


Fig. 20

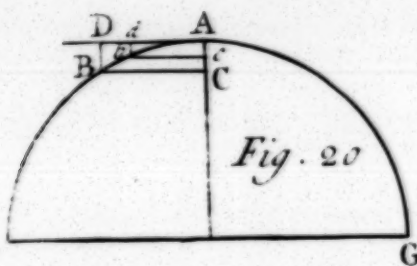


Fig. 21

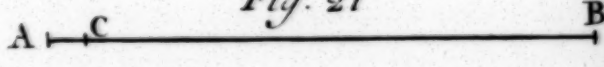


Fig. 22

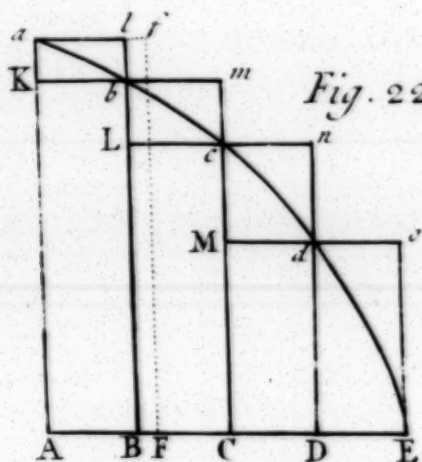


Fig. 23

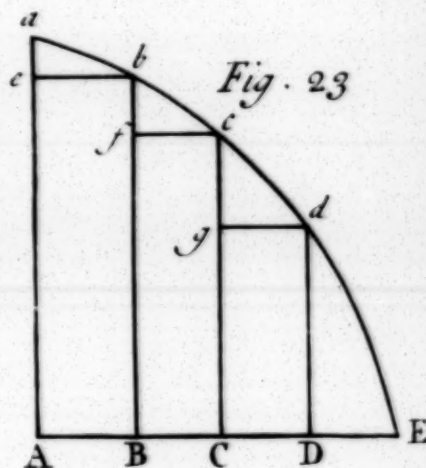


Fig. 24

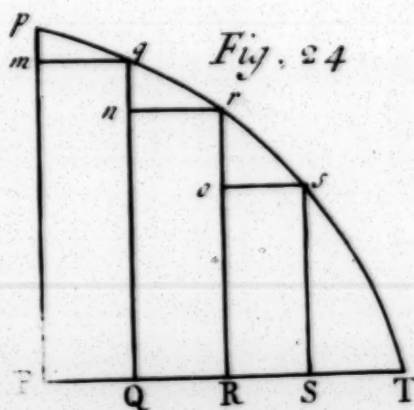
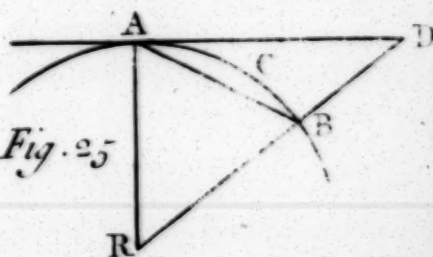
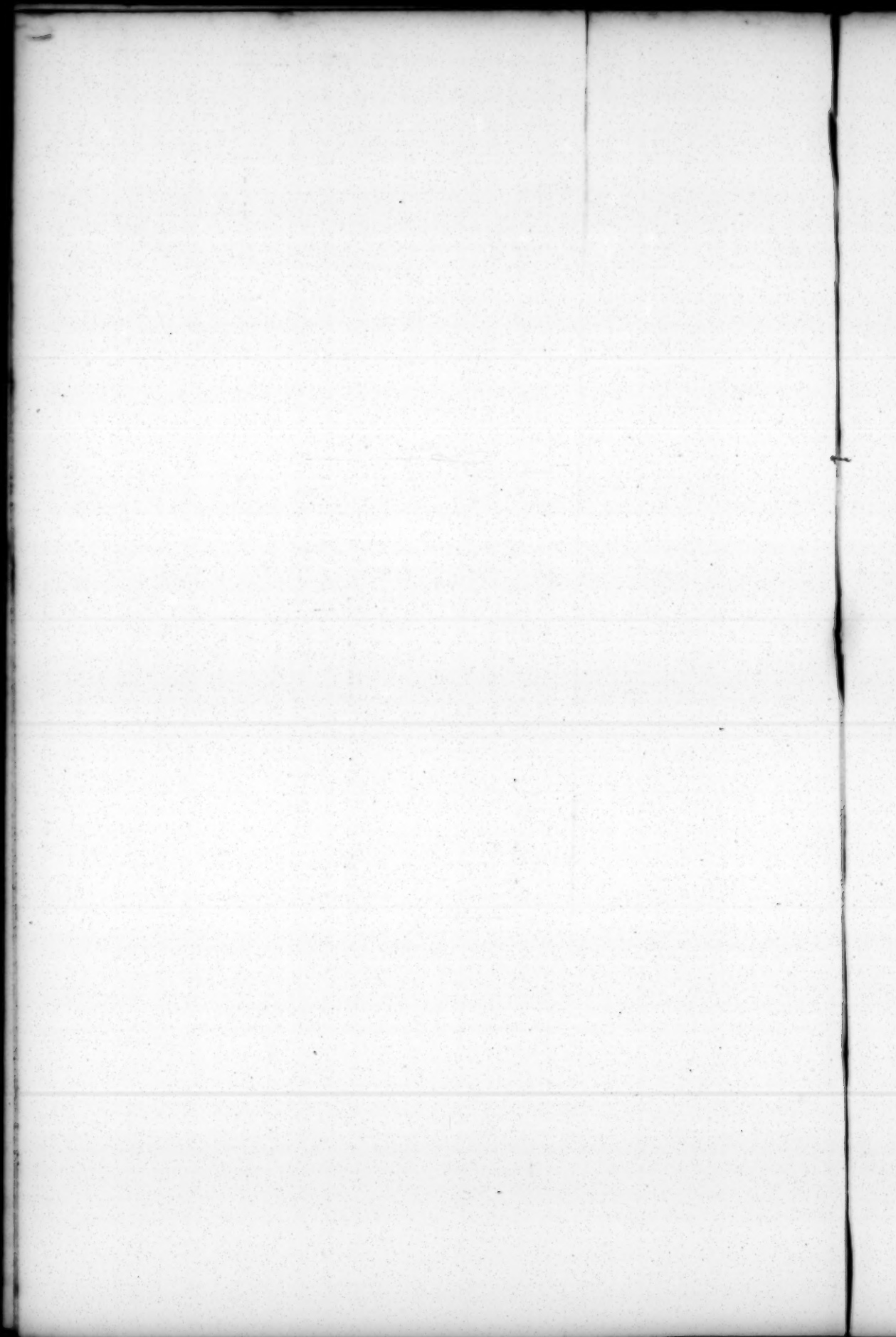


Fig. 25





similar to the streight Lines Ab , Ad , and the Arch $Ac b$) will vanish also, and that their last Proportion will be that of Equality. W. W. D.

Corol. 1. Hence it follows, that if through Fig. 27. the point B the Line BF be drawn parallel to the Tangent AD, intersecting continually any streight Line AF which passes through A; this Line BF will at last be equal to the vanishing Arch ACB; because if the Parallelogram AFB D be completed, BF will always be equal to AD.

Corol. 2. And if through the points B and A Fig. 27. there be drawn several streight Lines, as BE, BD, AF, AG, intersecting the Tangent AD and the Line BF which is parallel to it; the last proportion of all the Abscisses AD, AE, BF, BG, and of the Chord AB, and the Arch ACB, will be that of Equality.

Corol. 3. Wherefore all these Lines may indifferently be put for each other, in all our Reasonings about last Proportions.

L E M M A VIII.

If by two given streight Lines AR, BR, together with the Arch ACB, the Chord AB and the Tangent AD, there be formed three Triangles RAB, RACB, RAD; and then afterwards the Points A and B approach towards each other; I say, that the vanishing Triangles are similar to each other in their last Form, and that their last Proportion is that of Equality.

G

FOR

Concerning the MOTION of BODIES.

FOR let us imagine that during the time that B is approaching to A, the Lines AB, AD, AR, are always produced to the Points b, d, r at a good distance, and that $rb d$ is drawn parallel to RD, and the Arch $Ac b$ always similar to the Arch ACB. Then when the Points A and B coincide, the Angle bAd will vanish, and consequently the three Triangles which are always finite, $rAb, rAc b, rAd$ will coincide, and for that Reason become similar, and equal. Wherefore the Triangles RAB, RACB, RAD which are, by the Hypothesis, similar, and proportional to the Triangles $rAb, rAc b, rAd$, will at last be similar and equal to each other. W. W. D.

Corol. HENCE it follows, that these Triangles may be put indifferently for each other in all our Arguings about last Proportions.

L E M M A IX.

Fig. 28, 29. *If the streight Line AE and the Curve ABC, both given in Position, intersect each other in a given Angle A, and if the Ordinates BD, CE, be applied to the streight Line AE in another given Angle, and meet the Curve in BC; and if after this, the Points B and C arrive together at the Point A; I say, that the Areas of the Triangles ABD, ACE, will be to each other at last in a duplicate Proportion of their corresponding Sides, that is, as DA square to EA square, or as DB square to EC square, or as AB square to AC square.*

FOR

FOR let us imagine that during the time that the Points B and C are approaching to the Point A, the Line AD is always produced to the distant Points d and e in such a manner, that Ad and Ae shall be proportional to AD , and AE ; and let the Ordinates db, ec be erected parallel to the Ordinates DB, EC , and meet AB, AC produced, in b and c . Then imagine the Curve Abc to be drawn similar to the Curve ABC , and the straight Line Ag to be drawn so as to touch both the Curves in A , and to intersect the Ordinate applicates DB, EC, db, ec in the Points F, G, f, g ; then the Distance Ae continuing the same, let the Points B and C coincide with the Point A; and when the Angle cAg vanishes, the curvilinear Areas Abd, Ace will coincide with the rectilinear Areas Afd, Age , and therefore by (*Lemma V.*) they will be to each other in a duplicate Proportion of the Sides Ad, Ae . But the Areas ABD, ACE , are always proportional to the Areas Abd, Ace , and the Sides AD, AE are always proportional to the Sides Ad, Ae (*by the Hypothesis*;) Therefore the Areas ABD, ACE are at last in a duplicate proportion of the Sides AD, AE . W. W. D.

LEMMA X.

The Spaces described by a Body which is acted upon by any finite Force whatsoever, whether it be a determinate unalterable Force, or whether it continually increases or continually decreases

creases, are at the beginning of the Motion in a duplicate Proportion of the Times.

Fig. 29. LET the Times be represented by the Lines AD, AE, and let the Velocities generated in these Times be represented by the Ordinates DB, EC; then (because all Spaces described are as the Velocities multiplied by the Times, that is, as the Rectangles $AD \times DB$, $AE \times E$, or as half those Rectangles; that is, the Triangles ADB, AEC, if the Forces act uniformly, or as the trilinear similar Spaces ADB, ACE, if the Forces be proportional;) the Spaces described by these Velocities will be as the Arcas ABD, ACE described by these Ordinates; that is, in the beginning of the Motion, they will be as the Squares of the Times AD, AE (by *Lemma IX*,) W. W. D.

THUS if two Bodies be let fall together at the same Distance from the Center of the Earth, and the Space which one of them has fallen through in a single Moment of Time by the sole Force of Gravity, be compared with the Space which the other has fallen through in two Moments of Time by the same Force; these Spaces will be to each other as one to four, that is, if one of them has descended a Yard, the other will have descended four Yards.

LET us suppose two other equal Forces (besides the Force of Gravity) to act in the same manner upon these two Bodies, and to cause them to descend lower than they would have done by the single Force of Gravity, one Inch
suppose

suppose in the first Moment of Time; then the Differences of their Descents, or the Errors caused by these equal Forces, must be also as the Squares of the Times very nearly, that is, if one of the Bodies falls an Inch above a Yard in the first Moment of Time, the other Body will fall very nearly four Inches above four Yards in the two first Moments of Time. For,

Corol. I. FROM what has been said, it is easy to collect, that if Bodies describe similar parts of similar Figures in different Times, the Irregularities or Errors which are produced by any equal Forces whatsoever acting upon these Bodies in a similar manner, and which are measured by the Distances of the Bodies from those places of the similar Figures to which the same Bodies would arrive without those Forces, are in a duplicate Proportion of the Times in which they are produced, very nearly.

Let ADb, ade , be similar parts of similar Figures, described in the different Times T, t , by Forces compounded of the Forces of Gravitation, by which the Bodies A, a , would revolve uniformly in the Circles ABA, aca , and two other equal Forces F, F acting upon these Bodies in a similar manner in the same Times, by which equal Forces they are turned out of the Circles and describe other Curves Ab, ae : The Effects produced by these equal Forces are the Spaces Bb, ce , which are the Distances from the Points B and c which the Bodies would have arrived at by the Force of Gravitation only;

creases, are at the beginning of the Motion in a duplicate Proportion of the Times.

Fig. 29. LET the Times be represented by the Lines AD, AE, and let the Velocities generated in these Times be represented by the Ordinates DB, EC; then (because all Spaces described are as the Velocities multiplied by the Times, that is, as the Rectangles $AD \times DB$, $AE \times E$, or as half those Rectangles; that is, the Triangles ADB, AEC, if the Forces act uniformly, or as the trilinear similar Spaces ADB, ACE, if the Forces be proportional;) the Spaces described by these Velocities will be as the Areas ABD, ACE described by these Ordinates; that is, in the beginning of the Motion, they will be as the Squares of the Times AD, AE (by *Lemma IX.*) W. W. D.

THUS if two Bodies be let fall together at the same Distance from the Center of the Earth, and the Space which one of them has fallen through in a single Moment of Time by the sole Force of Gravity, be compared with the Space which the other has fallen through in two Moments of Time by the same Force; these Spaces will be to each other as one to four, that is, if one of them has descended a Yard, the other will have descended four Yards.

LET us suppose two other equal Forces (besides the Force of Gravity) to act in the same manner upon these two Bodies, and to cause them to descend lower than they would have done by the single Force of Gravity, one Inch
suppose

suppose in the first Moment of Time; then the Differences of their Descents, or the Errors caused by these equal Forces, must be also as the Squares of the Times very nearly, that is, if one of the Bodies falls an Inch above a Yard in the first Moment of Time, the other Body will fall very nearly four Inches above four Yards in the two first Moments of Time. For,

Corol. 1. FROM what has been said, it is easy to collect, that if Bodies describe similar parts of similar Figures in different Times, the Irregularities or Errors which are produced by any equal Forces whatsoever acting upon these Bodies in a similar manner, and which are measured by the Distances of the Bodies from those places of the similar Figures to which the same Bodies would arrive without those Forces, are in a duplicate Proportion of the Times in which they are produced, very nearly.

Let ADb, ade , be similar parts of similar Figures, described in the different Times T, t , by Forces compounded of the Forces of Gravitation, by which the Bodies A, a , would revolve uniformly in the Circles ABA, aca , and two other equal Forces F, F acting upon these Bodies in a similar manner in the same Times, by which equal Forces they are turned out of the Circles and describe other Curves Ab, ae : The Effects produced by these equal Forces are the Spaces Bb, ce , which are the Distances from the Points B and c which the Bodies would have arrived at by the Force of Gravitation only;

then will these Spaces Bb, ec , be, to each other as the Squares of the Times in which they are described, very nearly. Let E and e represent the Spaces Bb, ec , or the Errors produced by the equal Forces F, f , in the the different Times T, t , and which are measured by the Distances of the two Bodies from those Places which they would arrive at (by the Force of Gravity) without these equal Forces; then E will be to e as T^2 to t^2 exactly (*by this Lemma*;) but the Spaces DB, dc , through which these Bodies move by the Force of Gravity, considered distinctly from these Errors, are also as the Squares of the Times in which they are described; therefore when these Forces act jointly, the Errors will differ but infinitely little from the Squares of the Times in which their Spaces are described.

Corol. 2. IF the Forces by which the Errors are produced in Bodies describing similar parts of similar Figures, be different; and if these Forces act upon the Bodies in a similar manner, then the Errors will be in a Proportion compounded of the Forces and the Squares of the Times, that is, $E : e :: F \times T^2 : f \times t^2$.

Corol. 3. THE same thing is to be understood of any sort of Spaces whatsoever which are described by Bodies urged with different Forces: These Spaces will always be in the beginning of their Motions, in a Proportion compounded of the Forces and the Squares of the Times, that is, if S and s be the Spaces described, it will be, $S : s :: F \times T^2 : f \times t^2$.

Corol.

Corol. 4. WHENCE it follows, that at the beginning of the Times, the Forces are as the Spaces described directly and the Squares of the Times inversely; for if $S : s :: F \times T^2 : f \times t^2$. by equal Division, it will be $\frac{S}{T^2} : \frac{s}{t^2} :: F : f$.

Corol. 5. IN like manner, the Squares of the Times, are as the Spaces described directly and the Forces inversely, that is (by Transposition) $T^2 : t^2 :: \frac{S}{F} : \frac{s}{f}$.

SCHOLIION.

IF indeterminate Quantities of different Kinds be compared with one another, and if any one of them be said to be *as* any of the other directly or inversely: The Meaning is, that the first Quantity is increased or diminished in the same Proportion as the latter Quantity or its Reciprocal. Thus if A and B be two different sorts of Quantities continually increasing or decreasing, and it is said, that A is always *as* B directly, the Meaning is, that when A becomes twice or thrice &c. as big, or twice or thrice &c. as little as it was; B will also become twice or thrice &c. as big, or twice or thrice &c. as little as it was. If it be said, that A is *as* B inversely, the meaning is, that when A becomes twice or thrice &c. as big as it was, B will become twice or thrice &c. as little as it was; or when A becomes twice or thrice &c. as little as it was, B will become twice or thrice &c.

as big as it was. If any of the fore-mentioned Quantities be said to be *as* any two or more other Quantities directly or inversely; the Meaning is, that the first Quantity is increased or diminished in a Proportion compounded of the Proportions in which the other Quantities are increased or diminished; or in which their Reciprocals are increased or diminished. Thus if it be said that A is *as* B directly, and C directly, and D inversely, the Meaning is, that when A becomes any number of times bigger or less than it is, the compound Quantity $B \times C \times \frac{1}{D}$ will become so many times greater or less than it is likewise; or, which is the same thing, the Quantities A and $\frac{BC}{D}$ will be to each other in a given Proportion, or their Proportion to each other will always remain the same.

L E M M A XI.

The vanishing Subtense of the Angle of Contact, in all Curves which have a finite Curvature at the Point of Contact, is at last in a duplicate Proportion of the conterminat Arch.

THE Reason of adding these Words, *in all Curves which have a finite Curvature at the Point of Contact*, is because there are an infinite Number of Curves the Radius's of whose Curvatures are infinitely small, and such Curves are not here consider'd; but in all the *Conick Sections*, that is, in the Ellipsis, Parabola, and Hyper-

Hyperbola, if we imagine them to be composed of infinitely small Parts or elementary Curves; every one of these infinitely small Curves will be an infinite small part of a Circle whose Radius is a finite Quantity; that is, a Circle may be drawn through any Point in any Conick Section, whose Curvature is the same as the Curvature of the Section in that Point. (See *Milns's Conick Sections*, Par. III. Prop. X. XI.) It is to such sort of Curves as these latter only that this *Lemma* relates.

Case I. LET AB be that vanishing Arch, Fig. 31. AD its Tangent, BD the Subtense of the Angle of Contact perpendicular to the Tangent, and AB the Subtense or Chord of the Arch. Let AG, BG be erected perpendicular to the Tangent AD and the Subtense AB, and meet each other in G; then let the Points D, B, G come to the Points *d, b, g*, and let I be the Point where the Lines BG, AG intersect each other at last when the Points D and B coincide with the Point A. It is evident that the Distance Ig may become less than any assignable Distance whatsoever. Now if we suppose two Circles to pass through the Points ABG, Abg; AB^2 will be equal to $AG \times BD$, and $A b^2$ will be equal to $Ag \times bd$ (by the Property of the Circle;) wherefore the Proportion of AB^2 to $A b^2$ is compounded of the Proportions of AG to Ag, and BD to bd, that is, $AB^2 : A b^2 :: AG \times BD : Ag \times bd$. Now because GI may be taken less than any assignable Quantity whatsoever, the Proportion which AG bears to Ag, may

may also be nearer to Equality than can be represented by any Difference how small soever, and consequently the Proportion which AB^2 bears to $A b^2$ is nearer to the Proportion which BD bears to $b d$, than can be represented by any assignable Difference whatsoever. Wherefore (by *Lemma I*) the last Proportion of AB^2 to $A b^2$ is the very same as the last Proportion of BD to $b d$, that is, $AB^2 : A b^2 :: BD : b d$.

Case II. LET BD be inclined to AD in any given Angle whatsoever, as ABD ; and the last Proportion of BD to $b d$, will always be the same as before. For from the Points B, b , let the Lines $BE, b e$, be let fall to the Tangent AD , then because BD is parallel to $b d$, the Triangles $e b d, E B D$ are similar, consequently DB is proportional to EB or AC , and db is proportional to eb or Ac ; and therefore the Proportion of AB^2 to $A b^2$, which was as DB to db in the former Case, is also as DB to db in this Case.

Case 3. IF the Angle D be not given, so that the Line BD varies its Inclination to the Tangent AD , and converges towards a given Point, or is governed by any other Rule whatsoever; the Angle d being governed by the same Rule, these Angles must always come nearer and nearer to Equality, and before they vanish they will be so near that their Difference cannot be expressed by any finite Quantity whatsoever; and therefore at last they must be equal, by *Lemma I.* consequently the Lines $BD, b d$ must bear

bear the same Proportion to each other as they do in the former Cases. W.W.D.

Corol. 1. HENCE it follows, that because the Tangents AD , Ad , the Arches AB , Ab , and their Sines BC , bc , are at last when they vanish equal to the Chords AB , Ab , (*Lemma VII. Cor. 2.*) therefore the Squares of these vanishing Tangents, Arches and Sines, are as the Subtenses BD , bd .

Corol. 2. THE Squares of these Tangents, Arches and Sines, when they vanish, are also as the versed Sines of those Arches, if they bisect the Arches, and converge towards a given Point. For these versed Sines are in the same Proportion to each other as the Subtenses BD , bd . Thus let BE , be , be two very small Fig. 33. Arches bisected by the versed Sines cA , CA , both directed towards the same Point S ; then because (as has been demonstrated in Case II.) $BD:bd::CA:cA$; it will be $DA^2:dA^2::CA:cA$. or $BA^2:bA^2::CA:cA$. or $BC^2:bc^2::CA:cA$.

Corol. 3. HENCE it follows also that the versed Sines cA , CA are as the Squares of the Fig. 33. Times in which Bodies describe the Arches bA , BA with given Velocities. For if the Velocities be given, that is, if they do not increase or decrease, the Spaces bA , BA described by those Bodies, will be as the Times in which they are described, and the Squares of the Spaces bA , BA , will be as the Squares of the Times; but the Squares of these Spaces are as the versed Sines cA , CA .

Times;

Fig. 33. *Corol. 4.* The right-lined Triangles ADB , Adb when they vanish, or in their last Form, are in a triplicate Proportion of the Sides AD , Ad , that is, they are as the Cubes of those Sides. For the Triangle ADB , is to the Triangle Adb , as the Parallelogram $ADBC$, is to the Parallelogram $Adbc$; that is, putting A and a for those Triangles, it will be, $A : a :: AD \times DB : Ad \times db$; but (by this *Lemma*) DB is as AD^2 , and db as Ad^2 ; substituting therefore the one for the other, it will be $A : a :: AD^3 : Ad^3$. These Triangles are also in a sesquialterate Proportion of the Sides DB , db ; for because DB is as AD^2 , therefore (by equal Extraction) $\sqrt{DB}$ is as AD ; so likewise $\sqrt{db}$ is as Ad , consequently if these Quantities be substituted for AD , Ad in the Proportion $A : a :: AD \times DB : Ad \times db$; it will be $A : a :: \sqrt{DB} \times DB : \sqrt{db} \times db$. I mean by a sesquialterate Proportion, the Subduplicate of the Triplicate, that is, the square Root of a Cube or $\sqrt{DB^3}$, which is compounded of the simple Proportion DB , and the subduplicate Proportion $\sqrt{DB}$.

Fig. 33. *Corol. 5.* AND because DB , db are at last parallel and in a duplicate Proportion of AD , Ad , therefore the vanishing curve-lined Areas ADB , Adb will be equal to two third parts of the right-lined Triangles ADB , Adb , and the Segments AB , ab , will be each of them one third part of the same Triangles. (See *L'Hospital's Conick Sections*, Book V. Prop. XIII.) consequently



ED

~~B~~



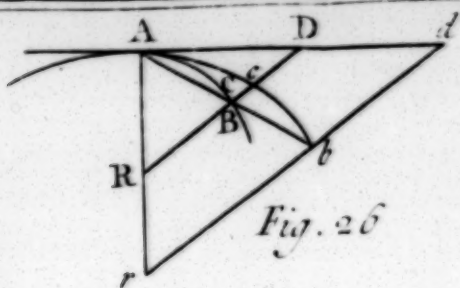


Fig. 26

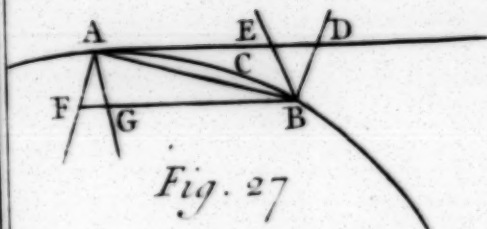


Fig. 27

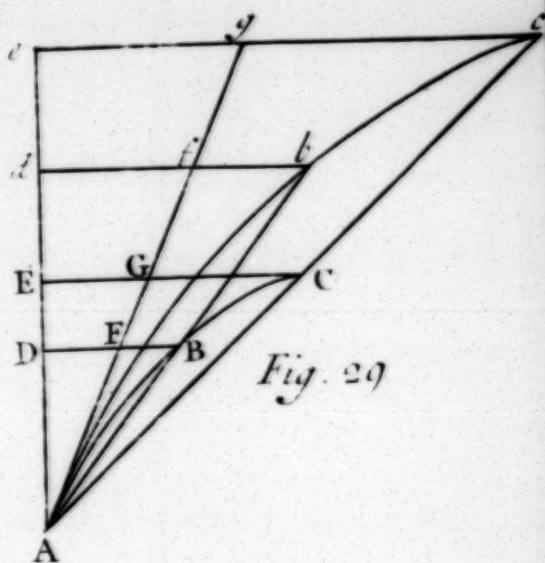


Fig. 29

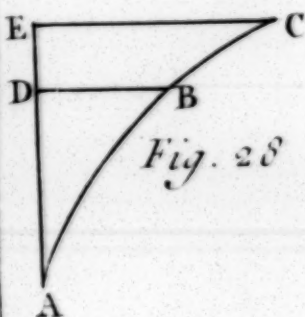


Fig. 28

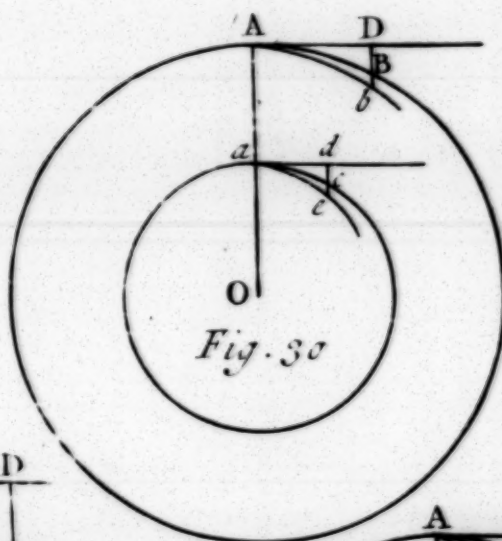


Fig. 30

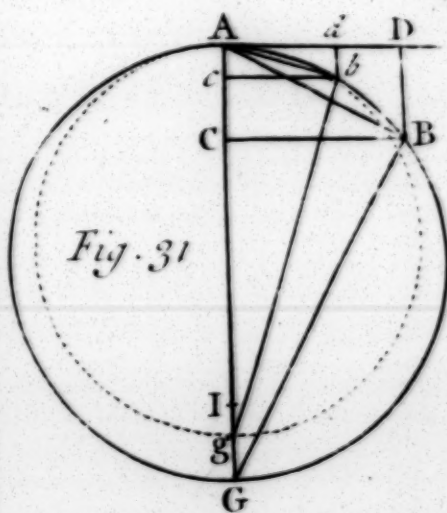


Fig. 31

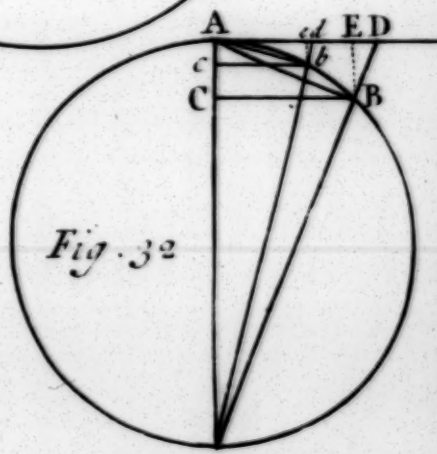


Fig. 32

8

7

quently these Areas and these Segments will be in a triplicate Proportion of the Tangents AD , Ad (by the foregoing *Corol.*) and also in a triplicate Proportion of the Chords and Arches AB , Ab , (by *Lemma VII. Cor. 2. 3.*)

SCHOLIION.

Now in all these Cases it is supposed, that the Angle of Contact DAB is neither infinitely greater nor infinitely less than the Angles of Contact which Circles contain with their Tangents; that is, the Curvature at the Point A is neither infinitely small nor infinitely great; or the Distance AI , the Diameter of the Circle Fig. 31. to which the Curvature of it is always reciprocally proportional, is of a finite Magnitude. For BD , which is the Measure of the Curvature of the Curve as well as the Diameter or Radius, may be taken proportional to AD cube, as well as in proportion to AD square; and in that Case no Circle can be drawn through the point A , which shall pass between the Tangent AD and that Curve AB ; for let the Radius of the Circle be of what length it will, the Curve will always pass between the Circle and the Tangent, and consequently, because an infinite Number of Circles may be drawn, and the same thing is true of them all, the Angle of Contact will be infinitely less than in Circles. And by the same way of Reasoning, if DB be successively in proportion to AD^4 , AD^5 , AD^6

AD^6 , AD^7 , &c. we shall have a Series of Angles of Contact going on infinitely, in which the latter is all along infinitely less than the former. And if DB be successively as AD^2 , $AD^{\frac{2}{3}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{3}}$, $AD^{\frac{6}{3}}$, $AD^{\frac{7}{3}}$, &c. we shall have another infinite Series of Angles of Contact, the first of which is of the same kind as the Angles of Contact in Circles, the second is infinitely greater, and in all the rest, every following one is infinitely greater than the foregoing one. So again there may be inserted, between any two of these Angles, a Series of intermediate Angles going on infinitely both ways, in which every succeeding one shall be infinitely greater or less than the foregoing one. Thus if between the Terms AD^2 and AD^3 , there be inserted the Series $AD^{\frac{1}{3}}$, $AD^{\frac{1}{2}}$, $AD^{\frac{2}{3}}$, $AD^{\frac{3}{4}}$, $AD^{\frac{4}{5}}$, $AD^{\frac{5}{6}}$, $AD^{\frac{6}{7}}$, $AD^{\frac{7}{8}}$, $AD^{\frac{8}{9}}$, $AD^{\frac{9}{10}}$, &c. And so again, between any two Angles of this Series, there may be inserted a new Series of intermediate Angles different from each other by infinite Distances: for Nature has no Bounds.

ALL that has been demonstrated concerning curve Lines and the Superficies contained in them, may easily be applied to the curved Superficies of Solids and the Contents of them. And I have premised these *Lemma's*, that I might avoid the tedious Method of reducing things to an Absurdity by long Demonstrations, according to the manner of the Antients. The Demonstrations indeed are made much shorter by the Method of Indivisibles. But because the

I

Hypo-

Hypothesis of Indivisibles is somewhat hard, and upon that account the Method is looked upon as less Geometrical; I therefore thought it a better Way to reduce the Demonstrations of the following Propositions to the last Sums and Proportions of vanishing Quantities, or to their first Sums and Proportions when they begin to be generated, that is, to the Limits of their Sums and Proportions, after which they become equal to nothing; and I have premised the Demonstration of these Limits with all the Brevity I possibly could. The same things are performed by this Method as by the Method of Indivisibles; and we may more safely depend upon the Principles demonstrated. Wherefore if at any time hereafter, I should consider Quantities as made up of Particles, or should make use of very small curve Lines instead of streight Lines, I would not be understood to mean Indivisibles, but vanishing Divisibles; not the Sums and Proportions of determinate or given Parts, but the Limits or Bounds of their Sums or Proportions always; and the Force of those Demonstrations are to be referred to the Method of the foregoing *Lemma's*.

PERHAPS it may be objected against what has been said, that there is no such thing as any last Proportion of vanishing Quantities, because before they are vanished, it is not the last Proportion, and when they are vanished there is no Proportion at all. But we may contend by the same Argument, that there is no such thing

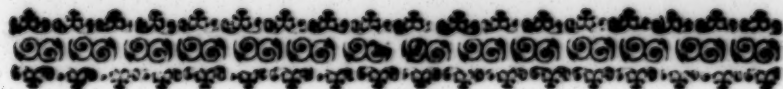
thing as the last Velocity with which a Body arrives at a certain place when its Motion is finished; for before the Body comes to this place, its Velocity is not the last, and when it is come to this place, it has no Velocity at all. But the Answer is easy: By the last Velocity, we are to understand the Velocity with which the Body moves, not before it arrives to the last place, nor the Velocity with which it moves afterwards, but the Velocity with which it arrives at that place, that is, the last Velocity with which the Body arrives at the last place, and with which the Motion ceases. So likewise, by the last Proportion of vanishing Quantities, we are to understand, not the Proportion of those Quantities before or after they are vanished, but the Proportion with which they vanish. For the same Reason, the first Proportion of Quantities that are generating, is the Proportion with which they begin to be generated. And the first and last Sum, is that with which they begin and end, that is, with which they begin to be increased or diminished, or with which they are finished. There is a certain Limit or Boundary which the Velocity can arrive at, in the end of the Motion, and beyond which it cannot go. This is the last Velocity. And the same thing holds good in the Limits of all Quantities and Proportions when they begin and end. And since this Limit is certain and determinate, it is a true Geometrical Problem to determine it; and any
I
thing

thing that is Geometrical may fairly be made use of in determining and demonstrating any other thing that is Geometrical also.

IT may also be urged, that if there be any such thing as the last Proportion of vanishing Quantities, there will be likewise last Magnitudes, and so every Quantity will be composed of Indivisibles, contrary to what *Euclid* has demonstrated in the tenth Book of his Elements concerning Incommensurables. But this Objection is built upon a false Hypothesis. The last Proportions with which Quantities vanish, are not the Proportions of last Quantities themselves, but they are the Limits towards which the Proportions of Quantities decreasing continually, are always approaching, and to which they can approach nearer than can be assigned by any given Difference whatsoever, and beyond which they can never go, nor can they arrive at them till the Quantities are infinitely diminished. This will be more plainly understood in the Case of infinitely great Quantities. If two Quantities, whose Difference is given, be increased infinitely, their last Proportion will be given, that is, they will be equal; but the last Quantities themselves, or the infinitely great Quantities to which the Proportion belongs, will not be given. If hereafter therefore, for the more easy Conception of things, I should at any time mention very small vanishing or last Quantities, I would by no means be understood to mean Quantities of any determined Bigness, but always such as decrease continually without any Bounds.

H

RULES



R U L E S

I N

NATURAL PHILOSOPHY.

R U L E I.

There ought to be no more Causes of natural Things allowed, than what are true and sufficient to explain their Phænomena by.

FOR Philosophers affirm, that Nature does nothing in vain or to no purpose; but it is to no purpose to effect any thing by a great many Ways which can be done by a few. For Nature is simple, and does not abound with a Multitude of superfluous Causes of Things.

R U L E II.

Wherefore the same Causes are to be assigned to all those Effects of natural Things which are of the same kind, so far as it can be done.

THUS

THUS Breathing, whether in Man or Beast, is to be ascribed to the same Cause; so is likewise the Descent of Stones when let fall, whether it be in *Europe* or in *America*: The Cause of Light ought to be looked upon as the same in a Kitchen-Fire as in the Sun, and the Cause of the Reflexion of Light here on the Earth, the same as in the Planets.

R U L E III.

Those Qualities of Bodies which can neither be diminished nor increased, and which are found in all Bodies which there can be any Experiments made in, ought to be looked upon as the Qualities of all Bodies whatsoever.

FOR we cannot come to the Knowledge of the Qualities of Bodies any other way but by Experiments; and therefore all those are to be esteemed general Qualities which agree to every Experiment that we can make. And those which cannot be at all diminished, cannot possibly be entirely taken away.

R U L E IV.

In experimental Philosophy, the Propositions gained from the Phænomena of Nature by an Induction of Particulars are to be looked upon as true, accurately or very nearly, notwithstanding any Hypotheses to the contrary,

till we meet with other Phænomena by which they may be rendered still more accurate, or else liable to Exceptions.

UNLESS this be allowed, an Argument built upon an Induction of Particulars may at any time be entirely destroyed by an Hypothesis.

TRUE Philosophy must be built upon Experiments, and those Experiments must be so many and so plain, that we cannot be deceived in them, but may argue universally from them, and apply them in all Cases and Circumstances whatsoever; for this is the only possible Way of coming at the Truth in these Things: Hypotheses are endless, and every Imagination of any ingenious Person may produce a new One; but the real Nature and Constitution of Things is Matter of Fact, and may be come at by constant Observations and repeated Trials. It is impossible by abstract Reasoning to find out the Quantity of Matter contained in the Planetary System, how many larger Parcels it is divided into, and what the particular Composition of those larger Parcels is; and it is equally impossible by such Reasoning to find out the particular Qualities of that Matter, and what Laws of Motion the several Parts or Particles of it are subject to. These are all of them originally merely arbitrary, and were at first chosen and prescribed by an intelligent Being, and are so ordered, that when the whole Composition

position was once framed and put together, it was capable of continuing for so long a Season as the wise Author of it intended. If therefore we would thoroughly understand the material System of the World, and see how it is held together, and what Laws of Motion the several Parts of it are subject to, we must do by it as we do by an artificial complicated Machine, take it to pieces, and examine well the Matter of which the several Parts are composed, and try by what Rules and Measures they are proportioned and adjusted to each other. Thus we may come to the Knowledge of the Qualities of Bodies by Experiments, and allow those Qualities to be general, which agree to all the Experiments that have ever been made; and where the Instances are such as cannot come under the Cognisance of our Senses, *viz.* in the very small Particles of Bodies, even here the Proportion must be owned to hold good, because Qualities which cannot be diminished, cannot be entirely taken away. This Method must surely be preferable to meer Dreams and Chimæra's, which have no Ground in Nature, but are directly contrary to the whole Procedure by Experiments; especially if we consider, that the Course of Nature is the shortest and most direct, and therefore we ought never to depart from the Analogy of it. It is by the Experience of our Senses that we come to know that any Bodies are extended; and there are some Bodies so small, that we are not sensible of their

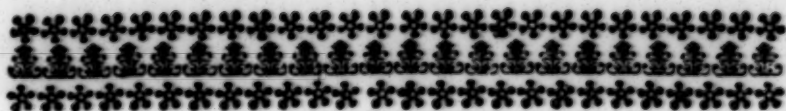
being extended; yet because we find all those that we can judge of by our Senses are extended, we justly conclude that all the other are so too. We find by Experience, that a great many Bodies are hard. Now the hardness of the whole must arise from the hardness of the Parts; and therefore we rightly argue that the small, and for ought we know, indivisible Particles, not only of those Bodies which come within the Reach of our Senses, but of all other Bodies also, are hard. That all Bodies are impenetrable, we gather by our Senses, and not by Reasoning; for all those that we feel upon, we find by Experience are impenetrable, and from thence we conclude that Impenetrability is the Property of all Bodies whatsoever. So likewise, that all Bodies are moveable, and continue of themselves, either in that State of Rest or Motion which they are once in, by some sort of Powers, (which may be called *inactive* Powers, or Powers that only keep things as they are) we gather from observing these Properties in those Bodies which we see. The Extension, Hardness, Impenetrability, Mobility and inactive Force of the whole Body, arise from the Extension, Hardness, Impenetrability, Mobility and inactive Force of the Parts; and from hence we conclude that every one of the smallest Parts of all Bodies is extended, and hard and impenetrable, and moveable, and endued with inactive Powers: And this is the Foundation of all Philosophy. Further, that the several distinct

distinct Parts, of which Bodies are composed, and which are contiguous with each other, may be separated from one another, we know by Experience; and it is certain from Mathematicks, that the Parts which are not actually divided, may in our Minds be distinguished into Parts still less. But whether those distinguished and to us undivided Parts, can really be divided or separated from each other by any Powers in Nature, is uncertain. But if it appeared by any single Experiment, that any hitherto undivided Particle could be broken off from a hard and solid Body, we should by the same Rule conclude, that not only the several distinct Parts were separable, but also that those which are not actually divided, might notwithstanding be divided into still less and less for ever. Further, if it appears from innumerable Experiments and astronomical Observations, that all Bodies here upon the Surface of the Earth are heavy, or do gravitate towards the Earth, in proportion to the Quantity of Matter contained in them; and that the Moon gravitates towards the Earth in proportion to the Quantity of Matter contained in it; and on the other hand, that our Sea gravitates towards the Moon, and that all the Planets gravitate towards each other, and that there is a like Gravitation in Comets towards the Sun, we may venture to affirm by the foregoing Rule, that all Bodies gravitate towards each other; because the Argument concerning universal Gravity, drawn from the Phæ-

nomena of Nature, is stronger than that concerning the Impenetrability of Bodies; for we can make no Experiments or Observations at all concerning this latter in the heavenly Bodies. However, we ought not to affirm that Gravitation is at all essential to Bodies; that Force, which is natural to all Bodies, is that inactive Power before mentioned, and this is unalterable; but Gravity is diminished as we go further from the Earth, in a certain Proportion, which shall be explained in the following Propositions.



SECT-



S E C T I O N II.

How to find out and measure

CENTRIPETAL FORCES.

PROPOSITION I. THEOREM I.

The Areas which Bodies moving in any Orb, describe by Rays drawn to the immoveable Center of their Forces, lie in the same Plane, and are proportionable to the Times in which they are described.

LET the Time be divided into equal Parts, and in the first part of the Time let the Body describe the streight Line AB by its own natural Force, or with the Velocity with which it would move on in a streight Line. In the second part of Time, if nothing hindered it, it would go streight on to *c*, describing the Line *cB* equal to AB (by *Law I.*) so that if the Rays AS, BS, *cS* be drawn, they would make the Triangles ASB, BS*c* equal to each other (by the *Elements*) because they are both of the same height. But when the Body comes to B, let

Fig. 34.

let the Centripetal Force, by one single but strong Impulse, cause the Body to go out of the streight Line Bc , and to go along in the streight Line BC . Let Cc be drawn parrallel to BS , and meet BC in C , then at the end of the second part of the Time, the Body will be found in C , in the same Plane with the Triangle ASB (by Corol. 1. of the Axioms or Laws of Motion.) Joyn SC , and the Triangle SBC will be equal to the Triangle SBc (by the Elements) because SB is parallel to Cc , and therefore the Triangle SBC will also be equal to the Triangle SAB . By the same Argument, if the Centripetal Force acts in the Points C, D, E succeßively, so as to make the Body describe the streight Lines CD, DE, EF , in so many equal parts of Time. These Lines will all lie in the same Plane (by Prop. 2. and 7th Book xi. of the Elements;) and the Triangle SCD will be equal to the Triangle SBC , and SDE to SCD and SEF to SDE . These small equal elementary Areas therefore, are described in equal Particles of Time, in an immoveable Plane; and consequently (by Composition) any Number of these Areas $SADS, SAFF$, are to each other, as the same Number of parts of Time, that is, as the Times in which these Areas are described. Let us now suppose the Time to be divided into infinitely small Particles; let us also suppose the Number of the Triangles to be increased infinitely, and their breadth to be diminished infinitely, then their sides $AB, BC, CD, \&c.$ will

will at last end in the Periphery ADF (by Corol. 4. Lem. III.) of some Curve: And therefore the centripetal Force with which a Body is continually drawn from the Tangent of such a Curve, acts perpetually upon it; and any Areas, such as SADS, SAFS, which are universally proportionable to the Times in which they are described, will in this Case also be proportionable to the Times. W. W. D.

Corol. 1. IN Spaces where there is no Resistance, the Velocity of a Body attracted towards an immoveable Center, is reciprocally as the Perpendicular let fall from that Center to the Tangent drawn from the Point of the Orb in which the Body is; for in equal parts of Time, the Velocity is as the Space described; therefore the Velocity in A, B, C, D, is as the Spaces AB, BC, CD, which are the Bases of so many equal Triangles ASB, BSC, &c. now these Bases are reciprocally as the Perpendiculars SZ, SY, &c. (by the Elements.)

Fig. 35.

Corol. 2. If the Chords AB, BC of two Arches described by the same Body immediately after each other, in equal Times, in Spaces where there is no Resistance, be compleated into the Parallelogram ABCV, and the Diagonal of it B-V be produced both ways, when it is in the Position it should be ultimately in when those Arches are infinitely diminished; this Diagonal will pass through the Center of the centripetal Forces S; for it is only an infinitely small part of the Ray SV.

Fig. 34.

Corol. 3.

Fig. 36. *Corol. 3.* IF the Chords AB, BC, and DE, EF, of Arches described in equal Times in Spaces where there is no Resistence, be compleated into the Parallelograms ABCV, DEFZ, the centripetal Forces in B and E, will be to each in the ultimate Proportion of the Diagonals BV, EZ, when those Arches are infinitely diminished. For the Motions of the Body, BC, EF, are compounded of the Motions Bc, BV, and Ef, EZ, (*by Corol. 1. of the Laws of Motion*) but BV and EZ are equal to Cc and Ff; and these are (*by the Demonstration of this Proposition,*) generated by the Impulses of the centripetal Force in B and E, and therefore are proportional to those Impulses (*by Definition VII.*)

Corol. 4. The Forces with which any Bodies moving in Spaces where there is no Resistence, are drawn from their rectilinear Motions, and turned into Curves, are to each other as the versed Sines $\frac{1}{2}$ VB, $\frac{1}{2}$ ZE, of the infinitely small Arches CA, FD, described in equal Times, which versed Sines converge towards the Center of the Forces S, and bisect the Chords CVA, FZD of those infinitely small Arches. For these versed Sines are half the Diagonals mentioned in the third *Corollary*, and Halves bear the same Proportion to each other, that their Wholes do.

Corol. 5. HENCE it follows also, that these centripetal Forces are to the Force of Gravity, as these versed Sines are to the versed Sines of the Arches of Parabola's described by projectile Bodies

Bodies in the same time, which latter versed Sines are perpendicular to the Horizon.

THUS let C be the Center of the Earth, or of the Curve described by the Body E revolving about C at the Distance CE; let ABD Fig. 37. be the Superficies of the Earth, and EB an Eminence upon it; from the Top of which E a Body is projected, and by the Force of Gravity falls to the Surface I in the parabolick Line EHI; then will the Centripetal Force be to the Force of Gravity as FG to FH (by this Corol.) for these are the versed Sines of the contemporary Arches EG, EH.

Corol. 6. The same things will all hold good (by Corol. 5. of the *Laws of Motion*) if the Planes in which the Bodies are moved do not continue at Rest, but move on also uniformly in a streight Line together with the Center of Forces which is in them.

PROPOSITION II. THEOREM II.

Every Body which is moved in any Curve Line described on a Plane, and a Ray being drawn from it to an immoveable Point, or to a Point which moves on uniformly in a streight Line, it describes Areas about such a Point proportionable to the Times; every such Body is urged by a centripetal Force towards that Point.

Case 1. FOR every Body which is moved in a Curve Line, is drawn from its rectilinear Course

Course by some Force acting upon it (*by Law the first.*) And that Force by which a Body is drawn from its rectilinear Course, and made to describe the very small
 Fig. 34. equal Triangles SAB, SBC, SCD, &c. in equal Times about the immoveable Point S, acts in the Point B according to the Direction of a Line parallel to cC (*by Prop. 40. Book I. of the Elements, and Law the 2d*) that is, according to the Direction of the Line BS; and in the Point C it acts in the Direction of a Line parallel to the Line dD , that is, in the Direction of the Line S, C, &c. Therefore the Force always acts according to the Direction of Lines tending to the immoveable Point S.

Case 2. AND it is the same thing (*by Corol. 5. of the Laws of Motion*) whether the Superficies in which the Body describes the curvilinear Figure, be at rest, or moved on uniformly in a streight Line, along with the Body, with the Figure described, and with the Point S belonging to it.

Corol. 1. IN Spaces or Mediums where there is no Resistence, if the Areas are not proportionable to the Times, the Forces do not tend to the Point where the Rays meet; but deviate from it, *in Consequence*, as Astronomers call it, that is, go the same way as the projected Body does, if the Description of the Areas be accelerated; or they deviate *in Antecedence*, or go the contrary Way to the projected Body, if the Description of the Areas be retarded.

Corol. 2. AND in resisting Mediums, if the Description of the Areas be accelerated, the Directions of the Forces must deviate from the Point where the Rays meet, and go the same Way as the Direction of the Motion of the projected Body.

THUS if the Description of the Areas be accelerated, let us suppose the Body instead of going from B to c (which it would do by its natural inherent Force, in the same Time that it went from A to B;) to go from B to l in the same Time. Then if the Parallelogram $dVB l$ be compleated, as before; and the Figure $sdlbs$, be made similar to the Figure $SCcBS$; 'tis manifest, (*by the Elements*) that the Triangle $s l B$ will be bigger than the Triangle $s l b$, which is equal to ScB , (*by the Construction*) and consequently the Description of the Areas will not be equable; but if lb be taken equal to cB , then the Triangle sdb , will be equal to the Triangle SCB , (*because the Figures $sdlbs$, $SCcBS$, are similar*) and described in the same Time. Wherefore if the Areas are not proportionable to the Times with respect to the Center S , but are so with respect to the Center s , it is a Sign that the Center S has moved out of its first place S into a new place s , in the same Time that it would have described the Line $B l$ by its natural or inherent Force; and this Motion of its Center is *in Consequence*, and would have been *in Antecedence*, had the point l been taken on the other side of c , as appears by the

Fig. 38.

Fig. 38.

Figure

Figure. So that in order to have a clear Notion how the Areas may not be proportionable to the Times in which they are described, but may be accelerated or retarded; it is only supposing the Triangle SBC to be moved out of its Situation, into the Situation sbd , with a Velocity proportional to the Velocity of the moveable Center, viz. proportional to Ss .

SCHOLIUM.

A BODY may be urged by a centripetal Force compounded of a great many Forces, and in that Case, the Sense of the Proposition is, that the whole Force which is compounded of all the Forces, tends to the Point S . Further, if any Force acts continually according to the Direction of a Line perpendicular to the Superficies described, such a Force will make the Body to go out of the Plane of its Motion, or cause it perpetually to alter the Plane of its Motion, but it will neither increase nor diminish the Quantity of the Superficies described, because it neither conspires with nor contradicts that Motion by which the Areas are generated, and therefore may be neglected in computing the Composition of Forces.

PROPOSITION III. THEOREM III.

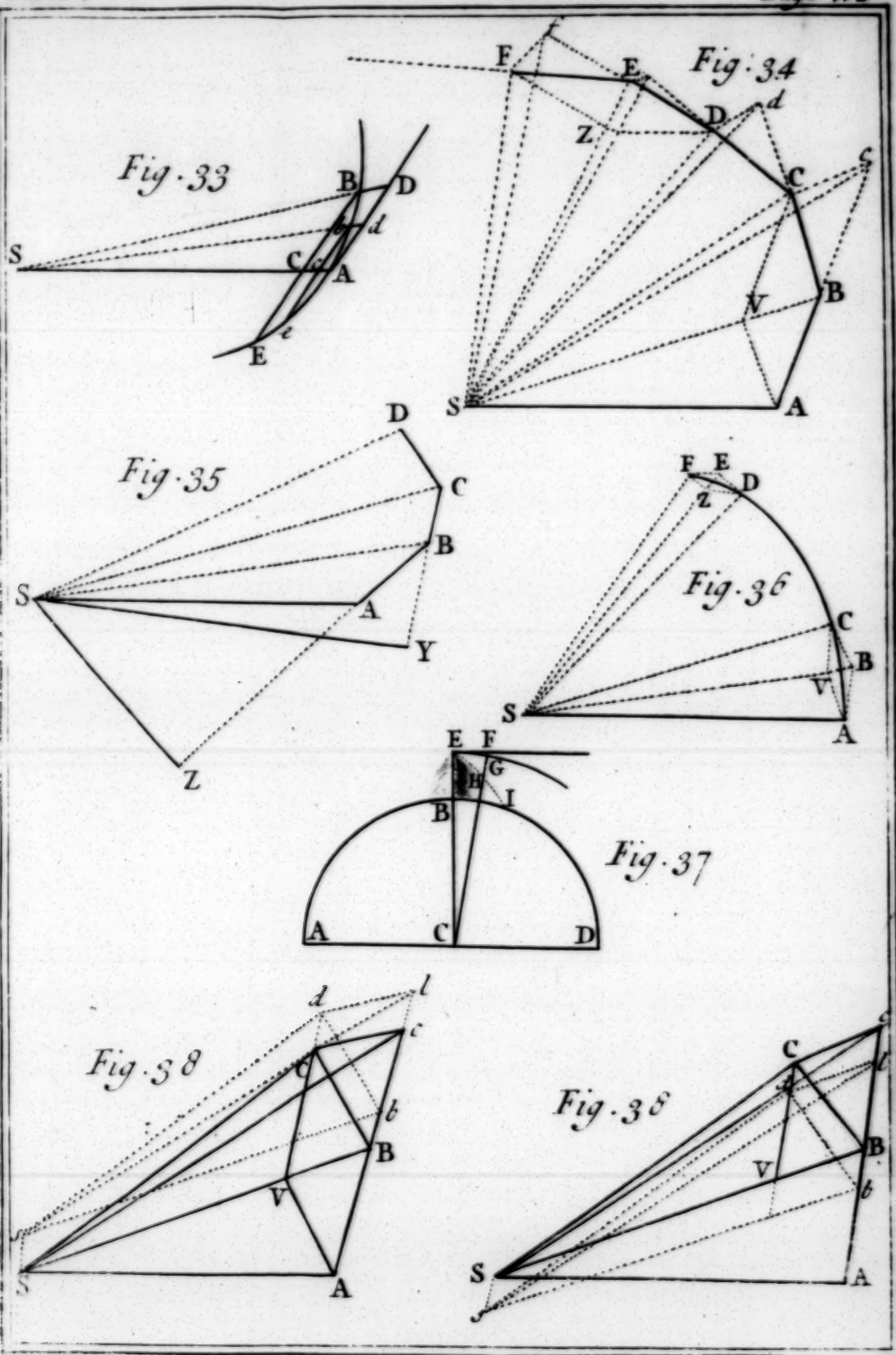
Every Body, which describes Areas proportionable to the Times by Rays drawn to the Center of another Body, which other Body is also moved
I
any



l
s
h
l
o
e
e
s
h
e
ts
d,
g

le
of
ed
ny

1



any way; is urged by a Force compounded of the centripetal Force tending to that other Body, and of the whole accelerative Force with which that other Body is moved.

LET L be the first Body, and let T be the other Body; and let L move about T in the Orb LQR , from L to R , and at the same time let the Center T be moved along in the Line TO , according to the Direction TO ; then if as the Body L moves from L to Q , both the Bodies L and T be urged along according to the Direction of the parallel Lines SN , OL , with a new Force which is equal, and contrary to that with which the Body T is moved from T to O , the Body L will notwithstanding go on to describe Areas proportional to the Times, about the other Body T ; (*by Cor. 6. of the Laws of Motion*) but the Force with which the other Body T was urged towards O , is entirely destroyed by this new Force, which is equal and contrary to it, and therefore (*by Law I.*) this latter Body T being now left to it self, will either continue at rest, or move on uniformly in a streight Line, and the first Body L being urged with the difference of the Forces, *viz.* QN , that is, being urged with the Force which remains, will continue to describe Areas proportional to the Times about the other Body T . The difference of the Forces therefore QN , tends to this latter Body T , as the Center $W.W.D.$

I

Corol.

Corol. 1. HENCE it follows, that if one
 Fig. 39. Body L describes about another Body T, by a
 Ray drawn from T to L, Areas proportional to
 the Times; and if from the whole Force (let
 it be a single Force, or let it be compounded of
 a great many Forces, *according to the second
 Corollary of the Laws of Motion*) with which
 the Body L is urged towards O, there be sub-
 ducted (*by the same Corol.*) the whole accelerative
 Force with which the other Body T is urged,
 Fig. 39. all the remaining Force, *viz.* QN, with which
 the first Body L is urged, tends to the latter
 Body T, as its Center.

Corol. 2. AND if those Areas are very nearly
 proportional to the Times; the remaining
 Force QN tends very nearly to the other
 Body T.

Corol. 3. AND on the other hand, if the re-
 Fig. 39. maining Force QN tends very nearly to the lat-
 ter Body T, the Areas will be very nearly pro-
 portional to the Times.

Corol. 4. IF the Body L, in moving about
 Fig. 39. another Body T, describes by a Ray drawn from
 L to T, Areas, which compared with the Times,
 are very unequal; and that other Body T is ei-
 ther at rest, or moves on uniformly in a streight
 Line: The Action of the centripetal Force
 tending towards the latter Body T, is either no-
 thing, or is mixed and compounded with the
 very powerful Actions of other Forces; and
 the whole Force compounded of all the Forces,
 if there be a great many, is directed to another
 Center

Center (either a moveable or an immoveable one.) And the same Argument holds good, what way soever the latter Body *T* be moved, if we take that centripetal Force *QN*, which remains after we have subtracted the whole Force which acts upon this latter Body *T*.

SCHOLIUM.

SINCE the equable Description of the Areas, discovers to us the Center to which this Force is directed, and this Force is that by which the Body is principally affected; being continually drawn by it from a streight Line, and kept in its Orb: why may we not in the following Propositions make use of this equable Description of the Areas, to find out the Center about which all circular Motion is performed in void Spaces? And is not the *Copernican* Hypothesis hereby confirmed and shown to be the true System of the Planets? For it appears by Observation, that the primary Planets do describe Areas proportional to the Times, by Rays drawn from the Earth to the Sun, and therefore the Sun is the Center of their Orbs; but they do not describe Areas proportional to the Times, if the Rays be drawn from the Earth to the Planets, consequently the Earth cannot be the Center of their Orbs. With respect to the Earth, they are sometimes Progressive, sometimes Stationary, and sometimes Retrograde: but with respect to the Sun, they are

always progressive, and go with an uniform Motion very nearly, but a little faster in their Perihelia, and a little slower in their Aphelia, so as to make the Description of the Areas equable. This is a Proposition well known to Astronomers, and is principally demonstrated in *Jupiter*, by the Eclipses of his Satellites, by which Eclipses the Heliocentrick Longitudes of that Planet, and his Distances from the Sun, are accurately determined. So likewise the Satellites of *Saturn* and *Jupiter* describe Areas proportional to the Times, by Rays drawn from those Planets to the respective Satellites; for it is manifest from Astronomical Observations, that the Orbs of these Satellites do not sensibly differ from Circles which are concentrick to *Saturn* and *Jupiter*, and their Motion in these Circles are always found to be progressive, so that the centripetal Forces must tend to the Bodies of these Planets. The Moon also describes an Area proportional to the Time, if there be a Ray drawn from it to the Center of the Earth; as is evident from comparing its apparent Motion with its apparent Diameter; and the last Proposition (*viz.* III.) is the Case of the Moon describing Areas proportional to the Time, as it moves about the Earth, at the same time that the Earth it self, to whose Center the Forces are directed, is carried together with the Moon about the Sun; and in this Case the centripetal Force of the Moon towards the

Fig. 39. Earth is always equal to QN , or the Difference
betwixt

betwixt the whole Force with which the Moon is moved, and the whole accelerative Force with which the Earth is moved about the Sun.

PROPOSITION IV. THEOREM IV.

If Bodies describe different Circles with an equable Motion or uniform Velocity; the centripetal Forces will tend to the Centers of those Circles, and will be to each other, as the Squares of the Arches described in the same Time, applied to the Radius's of the Circles; or they will be to each other as the Squares of the Arches described in the same Time, divided by the Radius's of the Circles.

LET BDB, bdb , be two different Circles, Fig. 40. in which two Bodies are moved with different, but uniform Velocities, then if the Time be divided into very small equal Parts, the uniform Velocities will be as the small Spaces run through in those parts of Time, that is, the Velocity of the Body moving in the Circumference BDB, will be to the Velocity of the Body moving in the Circumference bdb , as BD to bd (if these small Spaces be described in the same Time) or the Velocities will be to each other as any Number of these Spaces, viz. as $2BD$ to $2bd$, or as $3BD$ to $3bd$, and so on. Now because the Velocities in each Circle are uniform, all the small Spaces or Ele-

mentary parts BD of the Circle BDB , are equal to each other; and all the other elementary parts bd of the Circle bdb , are also equal to each other; therefore (*by the Elements*) all the infinitely small Areas or elementary Areas SDB , of which the whole Area of the Circle BDB is composed, are equal to each other, and all the elementary Areas sdb of which the whole Area of the Circle bdb is composed, are likewise equal to each other; consequently the centripetal Forces in each Circle tend to their common Center S (*by Prop. II. and Cor. 2. Prop. I.*) and are to each other as DC , dc , the versed Sines of the elementary Arches BD , bd , described in equal parts of Time (*by Cor. 4. Prop. I.*) that is, they are as the Squares of these Arches, applied to the Diameters of the Circles, or as $\frac{BD^2}{2BS}$ to $\frac{bd^2}{2BS}$. And because these Arches BD , bd , are to each other as any equal Number of the same Arches described in any equal Times (*by Composition*), and because the Radius of Circles are as their Diameters; Therefore (*by Substitution*) the centripetal Forces will be to each other as the Squares of any Arches described in the same Time, suppose $3BD$, $3bd$, applied to the Radius's of the Circles, or as $\frac{9BD^2}{R}$ to $\frac{9bd^2}{r}$, putting R , r , for BS , bs . W. W. D.

Otherwise

Otherwise thus :

THE very small Lines DC, dc , are as the Fig. 40. versed Sines of the Arches BD, bd , (by Cor. 2. Lem. VII.) described in the same time, and consequently they are as the centripetal Forces. Now (by Book III. Prop. 36 of the Elements,)

DC is equal to $\frac{BC^2}{BS+SC}$, and dc is equal to $\frac{bc^2}{bs+sc}$; but when D coincides with B, and d with b , then BS becomes equal to SC, and bs equal to sc ; and consequently DC becomes equal to $\frac{BC^2}{2BS}$ and dc equal to $\frac{bc^2}{2bs}$, that is, the same Proportion as before. W. W. D.

Corol. 1. SINCE the very small Arches BD, bd , are (by the Hypothesis) described in the same Time, the Velocities of the Bodies moved in these Circles will be as those Arches, that is, as the Spaces run through in a given Time. Therefore (by Substitution) if for the centripetal Forces in the Circles BDB, bdb , we put C, c , for the Radius's R, r , and for the Velocities V, v , the Proportion will be $C : c :: \frac{V^2}{R} : \frac{v^2}{r}$ that is, the centripetal Forces are in a duplicate Proportion of the Velocities directly, and a simple Proportion of the Radius's inversely.

Corol. 2. AND because the periodical Times are in a Proportion compounded of the direct Proportion of the Radius's, and the inverse

Proportion of the Velocities, (that is, are directly as the Spaces run through, and inversely as the Velocities,) therefore if for the periodical Times we put T, t , and in the Proportions of the preceding *Corollary* $\frac{V^2}{R}, \frac{v^2}{r}$ in-

stead of V^2, v^2 , we put $\frac{R^2}{T^2}, \frac{r^2}{t^2}$ and divide them

by R, r , it will be $C : c :: \frac{R}{T^2} : \frac{r}{t^2}$, that is,

the centripetal Forces are in a Proportion compounded of the direct Proportion of the Radius's, and the inverse Proportion of the Squares of the periodical Times.

Corol. 3. WHENCE it follows, that if the periodical Times T, t , are equal, and consequently $V^2 : v^2 :: R^2 : r^2$ or $V : v :: R : r$, then the Velocities are as the Radius's, and striking out the equal Quantities t^2, T^2 , in the foregoing *Corollary*, it will be $C : c :: R : r$, that is, the centripetal Forces will be as the Radius's. And on the contrary, if the centripetal Forces be as the Radius's, the Velocities will be also as the Radius's, and the periodical Times will be equal.

Fig. 41. For $BS \times DC$, is equal to BD^2 , and $bS \times dc$ is equal to $b d^2$ (by *Lem. XI.*) but DC and dc are as BS, bS (by the *Hypothesis*;) therefore BS^2 is as BD^2 , and bS^2 as $b d^2$, or (by equal Extraction, &c.) $BS : BD :: bS : b d$, that is, the Velocities are as the Radius's; and because BD and $b d$ are described in equal Particles of Time (by the *Hypothesis*) all the $BD's, b d's$, of which the whole Circles are composed, and which are equal

equal in number, will be described in equal whole Times, that is, the periodical Times will be equal. W. W. D.

Corol. 4. If the periodical Times be as the square Roots of the Radius's, or in a subduplicate Proportion of the Radius's; then if instead of T^2, t^2 , in the Proportions Cor. 2, we put R, r , it will be $V^2 : v^2 :: R : r$, or $V : v :: \sqrt{R} : \sqrt{r}$, that is, the Velocities will be in a subduplicate Proportion of the Radius's; and it will also be $C : c :: \frac{R}{\sqrt{R}} : \frac{r}{\sqrt{r}}$ that is, the centripetal Forces will be equal to each other. And the contrary.

Corol. 5. If the periodical Times be as the Radius's, and instead of T^2, t^2 , we put R^2, r^2 ; then it will be $V^2 : v^2 :: \frac{R^2}{R^2} : \frac{r^2}{r^2}$. that is, the Velocities will be equal; and it will also be $C : c :: \frac{R}{R^2} : \frac{r}{r^2}$ or $\frac{1}{R} : \frac{1}{r}$. that is, the centripetal Forces will be reciprocally as the Radius's. And the contrary.

Corol. 6. If the periodical Times be in a sesquialterate Proportion of the Radius's, that is, if the Squares of the periodical Times be as the Cubes of the Radius's, that is, if instead of T^2, t^2 , we substitute R^3, r^3 ; then it will be $V^2 : v^2 :: \frac{R^2}{R^3} : \frac{r^2}{r^3}$ or $V : v :: \frac{1}{\sqrt{R}} : \frac{1}{\sqrt{r}}$ that is, the Velocities will be reciprocally in a subduplicate Proportion of the Radius's, and it will also

also be $C : c :: \frac{R}{R^3} : \frac{r}{r^3}$ or $\frac{1}{R^2} : \frac{1}{r^2}$ that is, the centripetal Forces will be reciprocally as the Squares of the Radius's. And the contrary.

Corol. 7. AND universally, if the periodical Time be as any Power of the Radius's R^n, r^n , and instead of T^2, t^2 , in the fore-mentioned Proportions, we substitute R^{2n}, r^{2n} ; it will be

$$V^2 : v^2 :: \frac{R^2}{R^{2n}} : \frac{r^2}{r^{2n}} \text{ or } V : v :: \frac{R}{R^n} : \frac{r}{r^n} \text{ or } V : v ::$$

$\frac{1}{R^{n-1}} : \frac{1}{r^{n-1}}$ that is, the Velocities are reciprocally as the R^{n-1} th Power of the Radius's;

and it will be $C : c :: \frac{R}{R^{2n}} : \frac{r}{r^{2n}}$ or $C : c ::$

$\frac{1}{R^{2n-1}} : \frac{1}{r^{2n-1}}$ that is, the centripetal Forces are reciprocally as the R^{2n-1} th Power of the Radius's. And the contrary.

Fig. 42. Corol. 8. IF the Areas described in different Circles in the same Time be equal to each other, that is, if the Areas BSC , be equal to the Areas bSc , then (*by the Elements*) $BC : bc ::$

$\frac{1}{BS} : \frac{1}{bS}$ that is, the Velocities will be reciprocally as the Radius's: And (*by Composition*)

$BC^2 : bc^2 :: \frac{1}{BS^2} : \frac{1}{bS^2}$. Therefore if in the

Proportion $C : c :: \frac{BC^2}{BS} : \frac{bc^2}{bS}$ we put $\frac{1}{BS^2}, \frac{1}{bS^2}$

for BC^2, bc^2 ; it will be $C : c :: \frac{1}{BS^3} : \frac{1}{bS^3}$ that is,

is, the centripetal Forces will be reciprocally as the Cubes of the Radius's. And the contrary.

Corol. 9. THE shorter the Radius is, and the greater the Velocity, so much the wider will the Angle described in the same time be; that is, if for the Angle BSC, we substitute a , and for the Angle bSc , we substitute A ; then $A : a :: \frac{V}{r} : \frac{v}{R}$. and because

Fig. 42.

the Times are equal; therefore $V : v :: \frac{1}{r} : \frac{1}{R}$.

consequently (by Substitution) $A : a :: \frac{1}{r^2} : \frac{1}{R^2}$.

that is, the Angles described about the Center by Bodies moved in different Circles, are in equal Times reciprocally as the Squares of the Radius's.

Corol. 10. ALL the same things, concerning the Times, Velocities and Forces with which Bodies describe similar parts of any sort of similar Figures whatsoever, and whose Centers are in a similar Position in those Figures, follow from the Demonstration of the foregoing Particulars applied to these Cases. And they are to be applied by substituting the equable Description of the Areas for the equable Motion or given Velocity, and by making use of the Distances of the revolving Bodies from the Centers of their Forces instead of the Radius's. Thus suppose the Bodies were to move in similar Ellipses about their common Center, instead of moving in Circles, as in this Proposition: The Distances from the common Center

Center may be looked upon as the Radius's; and the momentaneous equal Areas may be put for the uniform Increase of Time, and the foregoing Demonstration will hold for finding out the centripetal Forces, periodical Times, Velocities, &c. Instances of which will come afterwards.

Corol. 11, IT follows also from the same Demonstration, that the Arch which a Body describes in any time whatsoever, by revolving uniformly in a Circle with a given centripetal Force, is a middle Proportion between the Diameter of the Circle, and the Space which the Body falls through by descending, with the same given centripetal Force, and in the same Time. For (*by the Demonstration of the Proposition*) BD^2 is equal to $2 BS \times DC$, or putting $2 BS$ equal to D the Diameter, then BD^2 is equal to $D \times DC$, consequently $D : BD :: BD : DC$. Now BD is the infinitely small Arch, described in an infinitely small part of Time, and DC is an infinitely small Space through which the Body has descended by the centripetal Force in the same Time; and (*by Composition*) the same Proportion will hold for any finite Arch whatsoever; that is, as BD is to DC , both generated in the same Moment of Time, so is any Arch whatsoever generated with the same uniform Velocity as BD , or any Arch composed of BD 's in any number of Moments of Time; to A Space generated with the same uniform Velocity as DC , or to a Space composed of the same Number

Number of DC's in the same Number of Moments of Time.

S C O L I U M.

FROM this last Proposition and its Corollaries we may learn the Nature of Gravitation, and the manner of its acting upon Bodies. It appears (*by Corol. 11.*) that the Force of Gravitation is less than any finite Force whatsoever, and yet notwithstanding, it is evident by the Proposition it self, that this same Force, as small as it is, decreases as the Square of the Distance from the Center; for however small the Arch BD be taken, and however small the part of Time in which it is described be; that is, however slow the Body moves, the Space DC described in the same part of Time, will be as much less than the Arch BD, as this Arch is less than the Diameter of the Circle, and consequently the Force of Gravitation, which is as DC, will be as much less than the Force with which the Body revolves in the Circle, which is as BD, as BD is less than 2 BS, the Diameter of the Circle. Thus if *Saturn* were to describe only an Inch in his Orb in a given Time; the Proportion would be; as the whole Diameter of his Orb, or as twice his Distance from the Sun, is to an Inch; so is his projectile Force, to his centripetal Force or Gravitation. The Forces of Gravitation therefore is of a quite different kind from the projectile Force, and cannot strictly be

be compared with it, but only by a compound Proportion made up of a finite and an infinite

Fig. 40. Proportion; as in this Proposition where BD^2 is equal to $DC \times 2BS$, or BD' is equal to $\sqrt{DC \times 2BS}$, that is, the projectile Force is in a subduplicate Proportion of the finite Quantity $2BS$, and the infinitely small Quantity DC . But though the centripetal Force be thus small compared with the projectile Force, yet by the last Corollary we may find how far and in how long Time a Body in descending, may by vertue of this Force, acquire a Velocity equal to that with which the Body revolves in the Circle; upon Supposition that the Action of the centripetal Force is uniform and constant, that is, the same at all Distances, and continually repeated; which Supposition is not indeed strictly true, because this Force (*by the Prop.*) decreases as the Squares of the Distances from the Center; but here near the Surface of the Earth, where the Distances of descending Bodies, compared with the Diameter of the Earth, are but small; the Errors will be insensible. Now this being allowed, *Galilæus*, *Hugenius*, and others, have demonstrated, that a Body would run through twice the length of any Space described, by falling in a given Time, with an uniform Velocity equal to that acquired at the end of its falling; which being granted, if in the Circumference of the Circle BDA , we take the Arch BD equal to the Radius BS , and in the Diameter BA , we take BC equal to a third

Fig. 43.

a third Proportional to BA, BD ; that is, if we take BC equal to half the Radius BS ; then (*by the last Corollary*) BC will be the Space described by the falling Body in the same time that the Arch BD is described by the revolving Body: But (*by the Hypothesis*) the falling Body, with the Velocity acquired in C , would run through twice the Space BC , in the same Time, that is, it would run through the Arch BD ; wherefore the Spaces described by the falling and by the revolving Body in the same Time being equal, the Velocities must be equal also. A Body therefore in falling through half the Radius or Distance from the Center, acquires a Velocity equal to that with which it revolves in the Circumference, or to its projectile Velocity.

HENCE it follows also, by way of Corollary, that if the centripetal Force acts uniformly, the Spaces described by falling Bodies will be as the Squares of the Times, and the Velocities acquired will be simply as the Times: For $BA \times BC = BD^2$, and $BA \times Bc = Bd^2$, Fig. 43. (*by the Hypothesis*;) but BA the Diameter is given, therefore BC is proportionable to BD^2 , and Bc is proportionable to Bd^2 , and because the Velocity with which the Body revolves in the Circle is equable, the Arches described are as the Times, and consequently BD^2 is proportionable to T^2 , and Bd^2 proportionable to t^2 : therefore $Bc:BC::t^2:T^2$, that is, the Spaces described are as the Squares of the Times. And because

the Acceleration is uniform or proportionable to the Times, the Velocities acquired will be as the Times also.

By such sort of Propositions as this, *Hugenius*, in his famous Treatise *de Horologio oscillatorio*, compared the Force of Gravity with the centrifugal Forces of revolving Bodies; and by the help of this Proposition and its Corollaries, we may find out the Proportion of the centripetal Force to any known Force, such as that of Gravity is. For if a Body were revolved in a Circle concentrick to the Earth, by the Force of its Gravity; this Gravity would be the very centripetal Force it self; and from the Descent of heavy Bodies the Time of one Revolution is given, and the Arch described in any given Time, is also given by Corol. 11. Thus suppose we would find out how long Time a Body would be in performing one Revolution at the Distance of the Superficies of the Earth from its Center; it appears by repeated Experiments, that a Body in falling describes fifteen *Paris* Feet and an Inch in one Second of Time, that is, in a sixtieth part of a Minute: It appears also from Experiments, that the Circumference of the Earth is 123249600 of the same Feet, and its Diameter 39231600. Now (*by the fore-mentioned Corol. 11.*) if we say, as 39231600, is to 123249600: so is 123249600 to 387148826. This third Proportional is the number of Feet that the Body will descend in the periodical Time;

I

but

but it descends fifteen Feet and an Inch in a Second of Time, and it was before shown that the Spaces it descends are as the Squares of the Times, wherefore by Computation, it will descend 387148826 Feet in one Hour, twenty four Minutes and twenty seven Seconds; and this is the periodical Time of the Revolution of a Body at the Distance of the Superficies of the Earth from its Center: In like manner any Arch described in a given Time may be found.

THE foregoing Particulars may also be demonstrated in this manner. In any Circles whatsoever ABCDE, *abcde*, let the similar Polygons AB, BC, CD, DE; *ab, bc, cd, de*, of any number of Sides whatsoever. Then if two equal homogeneous Bodies in moving along the sides of the Polygons with a given or equable Velocity, be reflected from the Circles at every one of the Angles, or at the Points A, B, C, D, E; *a, b, c, d, e*; the Forces with which they strike against the Circles will be as their Velocities; and therefore the Sums of their Forces in any given Time will be as their Velocities, and the Numbers of Reflections jointly or compounded together, that is, if *N, n*, be put for the Numbers of their Reflections, and *C, c*, for the Forces, it will be, $C:c::VN:vn$. But (if the Polygons be given in Species, or the Kind of them be determined,) the less the Polygon is, and the greater the length described in it

K in

in a given Time be, so many the more will the Number of Reflexions be; or the Numbers of Reflexions will be proportionable to the smallness of the Circles, and the length of the Arches of those Circles described in the same time, or as the Velocities directly, and the Radius's of the Circles inversely; that is, $N : n :: \frac{V}{R} : \frac{v}{r}$. therefore

(by Substitution) $C : c :: \frac{V^2}{R} : \frac{v^2}{r}$. that is, if the sides of the Polygons be infinitely diminished, so as to coincide with the Circles, the Forces will be to each other, as the Squares of the Velocities or Arches described in the same Time applied to the Radius's of the Circles. These are the centrifugal Forces with which the Bodies strike against the Circles, and the contrary Forces with which the Circles continually repel the Bodies towards their Center, is equal to them.

By this Proposition's being applied to the whole Planetary System, it appears likewise from the Phænomena, that the centripetal Force, and the Force of Gravitation in the heavenly Bodies, is the same, as well as here on the Earth; that it decreases as the Squares of the Distances, and is the single Principle by whose continual Energy all the Planets are kept in their respective Orbs about the Sun, as their common Center, and all
the

the Satellities are kept in their respective Orbs about their particular Planets, as their Centers. Thus

PHÆNOMENON I.

In the Satellities of Jupiter (supposing the fixed Stars to be at rest) their periodical Times are in a sesquialterate Proportion of their Distances from his Center, (as they ought to be by Corol. 6.)

THIS is allowed by all Astronomers, and is also manifest from the following Table.

The periodical Times of Jupiter's Satellities.

1^d. 18^h. 27'. 34". 3^d. 13^h. 13'. 42". 7^d. 3^h. 42'. 36".
16'. 16^h. 32'. 9".

The Distances of the Satellities from the Center of Jupiter are

By the Observations
of *Borallus*.

Townly by the Micro-
meter.

Cassini by the Tele-
scope.

Cassini by the Eclipses
of the Satellities.

By the periodical
Times.

1	2	3	4	Semidia- meters of Jupiter.
$5\frac{3}{4}$	$8\frac{3}{4}$	14	$24\frac{3}{4}$	
5, 51	8, 78	13, 47	24, 72	
5	8	13	23	
$5\frac{3}{4}$	9	$14\frac{3}{4}$	$25\frac{3}{4}$	
15, 667, 9, 017	14, 384	25, 299		

THE Elongations of the Satellities of Ju-
piter and his Diameter were determined by the
K 2 best

best Micrometers in the following manner. The greatest Heliocentrick Elongation of the fourth Satellity from the Center of *Jupiter*, was taken with a Tube of fifteen Feet long, and came out about $8'. 16''$. at the mean Distance of *Jupiter* from the Earth. The greatest Elongation of the third Satellity was taken with a Micrometer in a Telescope of 123 Feet long, and came out $4'. 42''$. The greatest Elongations of the rest of the Satellities, at the same Distance from the Earth, come out by the periodical Times $2'. 56''. 47''$. and $1'. 51''. 6''$.

THE Diameter of *Jupiter* was several times taken with a Micrometer, in a Telescope of 123 Feet long, and when reduced to the mean Distance of *Jupiter* from the Sun or the Earth, always came out less than $40''$. but never less than $38''$. very frequently $39''$. In shorter Telescopes, this Diameter is $40''$. or $41''$. For the Light of *Jupiter* is something dilated by the unequal Refrangibility, and this Dilatation bears a less Proportion to his Diameter in very long and more perfect Telescopes, than in shorter and less perfect ones. The Times in which two of the Satellities, the first and the third, passed over the Body of *Jupiter*, from the beginning of their entring into his Body, to the beginning of their coming out; and from their being entirely entered, to their being entirely come out, were observed with the same

same long Telescope. And the Diameter of *Jupiter*, at his mean Distance from the Earth, came out by the Passage of the *Satellity* $37''\frac{1}{4}$, and by the Passage of the third, it came out $37''\frac{3}{4}$. The Time in which the Shadow of the first *Satellity* passed over the Body of *Jupiter*, was also observed, and by that means the Diameter of *Jupiter*, at his mean Distance from the Earth, came out about $37''$. Let us take his Diameter at $37''\frac{1}{4}$, very nearly, and the greatest Elongations of the first, second, third and fourth *Satellites* will be equal to 5,965; 9,494; 15,141; and 26,63 Semidiameters of *Jupiter* respectively.

PHÆNOMENON II.

In the Satellities of Saturn (supposing the fixed Stars to be at rest) their periodical Times are in a sesquialterate Proportion of their Distances from his Center.

FOR *Cassinus* has by his own Observations determined their Distances from the Center of *Saturn*, and their periodical Times to be as follow.

The periodical Times of the Satellities of Saturn.

$1^h.21^h.18'.27''.2^d.17^h.41'.22''.4^d.12^h.25'.12''.$
 $15^d.22^h.41'.14''.79^d.7^h.48'.00''.$

K 3

The

The Distances of the Satellities from the Center of Saturn, in Semidiameters of his Ring.

By Observations,	$1\frac{1}{2}$.	$2\frac{1}{2}$.	$3\frac{1}{2}$.	8.	24.
By the periodical Times.	1,93.	2,47.	3,45.	8.	23, 35.

THE greatest Elongation of the fourth Satellicity from the Center of *Saturn*, uses to be computed at very near eight Semidiameters. But the greatest Elongation of this Satellicity from the Center of *Saturn*, taken by one of the best Micrometers in Mr. *Hugenius's* Telescope of 123 Feet long, came out eight Semidiameters and seven tenth parts of a Semidiameter. Now by this Observation and the periodical Times, the Distances of the Satellities from the Center of *Saturn*, in Semidiameters of his Ring, are 2, 1. 2, 69. 3, 75. 8, 7. and 25, 35. The Diameter of *Saturn*, by the same Telescope, was to the Diameter of the Ring, as 3 to 7, and the Diameter of the Ring in the 28th and 29th Days of *May* 1719, came out 43". whence the Diameter of the Ring, in the mean Distance of *Saturn* from the Earth, is 42", and the Diameter of *Saturn* himself 18". These things are thus in the longest and best Telescopes, because the apparent Magnitudes of the celestial Bodies, through very long Telescopes, bear

bear a greater Proportion to the Dilatation of the Light at the Extremities or Edges of those Bodies, than in shorter Telescopes. If all the false Light be rejected, the Diameter of *Saturn* would not remain larger than 16".

P H Æ N O M E N O N III.

The Orbs of the five primary Planets, Mercury, Venus, Mars, Jupiter and Saturn, surround the Sun.

THAT *Mercury* and *Venus* revolve about the Sun, is demonstrated, by their having the same Phases as the Moon. They appear full when they are beyond the Sun, and in their first and last Quarters when on each side of the Sun, and horned when on this side of the Sun, and they sometimes pass over his Disk like black Spots. It is also certain, from *Mars's* appearing full when he is nearly in Conjunction with the Sun, and gibbous in his Quadratures, that he moves about the Sun. And the same thing is demonstrably so in *Jupiter* and *Saturn*, because they appear always full: And it is manifest in these last, from the Shadows of the Satellities cast upon them, that they shine with Light borrowed from the Sun.

P H Æ N O M E N O N IV.

The periodical Times of the five primary Planets, and that of the Sun about the Earth, or the Earth about the Sun (if the fixed Stars be at rest) are in a sesquialterate Proportion of their mean Distances from the Sun.

THIS Proportion, first found out by *Kepler*, is acknowledged by every Body. And the periodical Times are the same, and the Dimensions of their Orbs are the same, whether the Sun revolves about the Earth, or the Earth about the Sun. And as to the length of the periodical Times, it is agreed amongst all Astronomers. But *Kepler* and *Bulialdus* have the most diligently of them all determined the Magnitudes of their Orbs from Observations: and their mean Distances which correspond to the periodical Times, do not sensibly differ from those Distances which they found out, but are for the most part mean Distances between the other, as you may see in the following Table.

The

The periodical Times of the Planets and Earth about the Sun, with respect to the fixed Stars, in Days and decimal parts of a Day.

h	u	f
10759,275.	4332,514.	686,9785.
d	q	p
365,2565.	224,6176.	87,9692.

The mean Distances of the Planets and Earth from the Sun.

According to Kepler,

h	u	f
951000.	519650.	152350.
d	q	p
100000.	72400.	38806.

According to Bulialdus,

954198.	522520.	152350.
100000.	72398.	38585.

According to the periodical Times,

944006.	520096.	152369.
100000.	72333.	38710.

THERE can be no Dispute about the Distances of *Mercury* and *Venus* from the Sun, because they are determined by their Elongations from the Sun; neither can there be any Ground for Dispute concerning the Distances of the superior Planets from the Sun, if we observe the Eclipses of the Satellities of *Jupiter*; for by these Eclipses the Position of the Shadow which *Jupiter's* Body casts

casts, is determined, and by that means his Heliocentrick Longitude is had; and from comparing his Heliocentrick and Geocentrick Longitudes with each other, the Distance of *Jupiter* is determined. This you may see demonstrated in Dr. *Keil's* Astronomical Lecture XVI.

HENCE the following Propositions may be deduced.

- I. *The Forces with which the Satellities of Jupiter are perpetually drawn from their Motions in streight Lines, and kept in their several Orbs, is reciprocally as the Squares of their Distances from the Center of Jupiter.*

THIS is evident from the first Phænomenon, and the sixth Corol. of this Proposition.

AND the same thing is to be understood of the Satellities of *Saturn*, by the same Corol. and the second Phænomenon.

- II. *The Forces with which the primary Planets are perpetually drawn from their Motions in streight Lines, and kept in their several Orbs, is reciprocally as the Squares of their Distances from the Sun.*

BY the same Corol. and the fourth Phænomenon.

III. *The*

III. *The Moon gravitates towards the Earth, and by the Force of its Gravitation, is perpetually drawn from its Motion in a streight Line, and kept in its Orb.*

THE mean Distance of the Moon from the Earth is in its Syzygies, according to *Ptolomy*, and most Astronomers, fifty nine Semidiameters of the Earth, according to *Vendelinus* and *Hugenius* sixty, according to *Corpernicus* sixty and one third part, according to *Street* sixty and two fifth parts, and according to *Tycho* fifty-six and a half. But *Tycho*, and all they that follow his Tables of Refractions, by making the Refractions of the Sun and Moon (directly contrary to the Nature of Light) greater than that of the fixed Stars by four or five Minutes, make the Moon's Parallax greater than it should be by so many Minutes, that is, about a twelfth or a fifteenth part of the whole Parallax. Let that Error be corrected, and the Distance will be sixty Semidiameters of the Earth and a half, which is pretty near what others have commonly made it. We will take the mean Distance in the Syzygies to be sixty Semidiameters; and the periodical Time, with respect to the fixed Stars, to be 27 Days, 7 Hours and 43 Minutes, as agreed on by Astronomers; and let the Circumference of the Earth be the same which
the

the *French* determined by measuring, *viz.* 123249600 *Paris* Feet. Then if we suppose the Moon's Motion to be entirely stopped; and that the Moon be let fall in such a manner, as to descend towards the Earth with all that Force with which it is retained in its Orb, it would in falling describe fifteen *Paris* Feet and an Inch in a Minute of Time. This may be computed from Corol. 11. exactly in the same manner as was just now computed, when we compared an entire Revolution of a Body, at the Distance of the Earth's Superficies, with Gravitation. For the versed Sine of an Arch described by the Moon in one Minute of Time, with his mean Motion at the Distance of sixty Semidiameters of the Earth, is about fifteen *Paris* Feet and an Inch, or more accurately, 15 Feet, 1 Inch and $1 \frac{4}{9}$ of a Line; whence it follows, because (*by this Prop.*) this Force, as it comes nearer the Earth, increases as the Square of the Distance decreases, and consequently is sixty times sixty greater at the Superficies of the Earth than at the Moon; hence, I say, it follows, that a Body falling here upon the Earth with that Force so increased, would describe sixty times sixty, multiplied by fifteen *Paris* Feet and one Inch, in one Minute of Time; or, because the Spaces described are as the Squares of the Times, it would describe fifteen *Paris* Feet and an Inch in a Second of Time; or, more accurately,

accurately, 15 *Paris* Feet, 1 Inch and $1\frac{4}{9}$ of a Line; so that a Body is a whole Minute in falling the same Space at the Distance of the Moon, that it falls in a Second here upon the Surface of the Earth. For in the Latitude of *Paris*, a Pendulum that vibrates Seconds, is three *Paris* Feet and $8\frac{1}{2}$ Lines long, as *Hugenius* observed. And the Space which a heavy Body describes by falling in a Second of Time, is to half the length of such a Pendulum; in a duplicate Proportion of the Circumference of the Circle, to the Diameter (as the same *Hugenius* has shewn) and is therefore 15 *Paris* Feet 1 Inch and $1\frac{7}{9}$ of a Line. Wherefore if the Moon were to descend to the Superficies of the Earth, the Force by which it is kept in its Orb would become equal to the Force of Gravity here; and therefore (*by Rule I. and II.*) is the very same Force which we commonly call Gravity. For if Gravity were something that is different from this Force, Bodies in descending towards the Earth with both these Forces united together, would move twice as fast, and in a Second of Time would in falling describe $30\frac{1}{2}$ *Paris* Feet, which is contrary to all Experience.

SCHOLIUM.

THE Demonstration of this last Proposition may be more largely explained in the follow-

following manner. If there were several Moons revolving about this Earth, as there is in the System of *Saturn* and *Jupiter*: their periodical Times would observe the same Law as that taken notice of by *Kepler* in the Planets (*by Induction.*) and therefore their centripetal Forces would be reciprocally as the Squares of their Distances from the Center of the Earth (*by the first of these Propositions.*) And if the nearest of these Moons were a small one, and very nearly touched the Tops of the highest Mountains; the centripetal Force by which it is retained in its Orb, would be very near equal to the Gravity of Bodies upon the Top of those Mountains (*by the foregoing Computation,*) and would cause that Moon, if it were deprived of all that Motion with which it goes along in its Orb, to descend towards the Earth, for want of that Centrifugal by which it is kept in its Orb; and that with the same Velocity with which heavy Bodies fall on the Tops of the same Mountains; because the Forces with which they descend are equal. For if the Force with which this nearest Moon descends, were different from Gravity; and this Moon were heavy besides, and tended towards the Earth, as all other Bodies on the Tops of these Mountains do; this Moon would descend twice as fast as they, because of both the Forces united together. Wherefore since these two Forces, *viz.* the Forces of heavy Bodies
I and

and the Force of these Moons do both tend to the Center of the Earth, and are of the same sort, and equal to each other, they must have the same Cause (*by Rules I. and II.*) Wherefore that Force by which the Moon is retained in its Orb, is that very Force which we commonly call Gravity; for if it were not so, the Moon now mentioned, near the Top of the Mountain, must be affirmed either to be without any Gravity at all, or allowed to fall twice as fast as any heavy Bodies do.

IV. *The Satellities of Jupiter gravitate towards Jupiter, the Satellities of Saturn gravitate towards Saturn, and the Planets gravitate towards the Sun, and are drawn from their rectilinear Motions, and kept in curvilinear Orbs by the Force of this Gravity.*

FOR the Revolutions of the Satellities of *Jupiter* about *Jupiter*, and the Revolutions of the Satellities of *Saturn* about *Saturn*, and the Revolutions of *Mercury* and *Venus*, and the rest of the Planets about the Sun, are Phænomena of the same kind as the Revolution of the Moon about the Earth; and therefore (*by Rule II.*) they must depend upon the same sort of Cause: especially since it has been demonstrated that the Forces upon which those Revolutions depend, tend to-
wards

wards the Centers of *Jupiter*, *Saturn*, and the Sun, and that in receding from those Centers, they decrease in the same Proportion, and according to the same Law, as the Force of Gravity does in receding from the Earth.

Corol. 1. THERE is therefore a Gravitation towards all the Planets. For no Body doubts but that *Venus* and *Mercury*, and the rest of them, are Bodies of the same kind with *Jupiter* and *Saturn*. And since all Attraction is mutual, (*by the third Law of Motion*;) *Jupiter* must gravitate towards all his Satellities, *Saturn* towards all his Satellities, the Earth must gravitate towards the Moon, and the Sun towards all the primary Planets.

Corol. 2. THE Gravity tending towards every Planet is reciprocally as the Square of the Distance from the Center.

Corol. 3. THE Planets all gravitate towards each other (*by Cor. 1. and 2;*) and hence it follows that *Jupiter* and *Saturn*, when they are nearly in Conjunction, must sensibly disturb one another's Motion, by their mutual Attraction; the Sun must also, for the same Reason, disturb the Moon's Motion, and the Sun and Moon disturb our Sea.

SCHOLIUM.

THE Force with which the Cælestial Bodies are retained in their Orbs, has been hitherto

therto called a centripetal Force; it is now manifest, that it is no other than the Force of Gravity, and therefore for the future we shall call it Gravity. For the Cause of that centripetal Force by which the Moon is retained in his Orb, ought to be extended to all the Planets, (by Rules 2. and 4.)

PROPOSITION V. PROBLEM I.

In any places whatsoever, the Velocity being given with which a Body describes a given Figure, with Forces tending to any common Center; it is required to find that Center.

LET the three streight Lines P T, T Q V, Fig. 45 V R, touch the Figure described in so many points P, Q, R, and meet each other in T and V. Let the Lines P A, Q B R C, be erected perpendicular to these Tangents, and let them be reciprocally proportionable to the Velocities of the Body in the Points P, Q, R, from whence they are erected; that is, let P A, be to P Q, as the Velocity in Q, is to the Velocity in P, and let Q B, be to R C, as the Velocity in R, to the Velocity in Q. Through the Terms or Ends of the Perpendiculars A, B, C, let A D, D B E, E C, be drawn at right Angles, and meet each other in D and E: Then, if the Lines T D, V E, be drawn and produced, they will meet in S, the Center required.

L

F O R

FOR the Perpendiculars SF, SH, SL, let fall from the Center S, upon the Tangents FT, TV, VL, are reciprocally as the Velocities of the Bodies in the Points P, Q, and R, (*by Cor. 1. Prop. I.*) and therefore (*by the Construction*) as the Perpendiculars AP, BQ, CR, directly; that is, as the Perpendiculars DG, DI, EK, let fall from the Points D and E, to the Tangents: Now because the Proportions of SF, to SH; DG, to DI; and IT, to GT, are the same, that is, are reciprocally as the Velocities in P and Q; therefore (*by the Elements*) SFT H, and GDIT, are similar Parallelograms, and SDT, a Diagonal; and consequently the Points S, D, T, lie in one streight Line. By the same Argument the Points S, E, V, lie in one streight Line also; and therefore the Center S is in the Point where the two streight Lines TD, VE, meet. W. W. D.

PROPOSITION VI. THEOREM V.

If in a Space where there is no Resistance, a Body be revolved in any Orb whatsoever, about an immoveable Center, and describes any incremental Arch, or Arch generated in a very small time, and the versed Sine of this Arch be conceived to be so drawn as to bisect its Chord, and being produced, passes through the Center of the Forces; the centripetal Force in the middle of this Arch

will be as the versed Sine directly, and the Square of the Time inversely.

LET OV be the incremental Arch, OL V, Fig. 46. its Chord, bisected in L; let the versed Sine LR be so drawn, that when produced, it may pass through S, the Center of the Forces, and let T be put for the Particle of Time in which the Arch OV is described: I say, that the centripetal Force C, in the Point R, is proportional to $\frac{LR}{T^2}$. For suppose a smaller Arch IV to be described in a given Time t , the versed Sine of this Arch, viz. QP; is as the centripetal Force C; (by Corol. 4. Prop. I.) but if the given Time t be increased in any Proportion whatsoever, the Arch IV will be increased in the same Proportion; that is, when t becomes equal to T; IV will become equal to OV, (because the Velocity being given, the Times are as the Spaces described;) And when the versed Sine QP becomes equal to LR, it increases as the Square of the Arch OR, or as the Square of the Time T (by Corol. 2. and 3. Lem. XI.) The versed Sine LR therefore is in the compound Proportion of the Force C, and the Square of the Time T; that is, $LR = C \times T^2$, or dividing both sides by T^2 , it will be $\frac{LR}{T^2} = C$. And the same Demonstration holds for any other incremental Arch ov . So that whatever Proportion

tion the Times in which the Arches OV , ov , are described, bear to each other (as T to t , suppose) the centripetal Forces in R and r will be to each other, as the versed Sines RL , rl , directly, and the Squares of those Times inversely; that is, $C : c :: LR : lr$. W. W. D.

THE same may also easily be demonstrated by Corol. 4. Lem. X. thus.

Fig. 46. IF the Forces in R and r be given, the Spaces described, viz. LR , lr , are as the Squares of the Times (*by that Lemma.*) but if those Forces be not given, then the Spaces described will be as the Squares of the Times and the Forces together; that is, LR will be as $C \times T^2$, and lr , as $c \times t^2$, the same Proportions as before. W. W. D.

Fig. 47. Corol. 1. If the Body P in revolving about the Center S , describes the curve Line APQ ; and if the straight Line ZPR touch this Curve in any point P , and from any other point of the Curve, as Q , there be drawn to the Tangent, the Line QR , parallel to the Distance SP , and from the same point Q , there be let fall QT perpendicular to the same Distance SP : the centripetal Force will be reciprocally as the Solid $\frac{SP^2 \times QT^2}{QR}$, if you always take the last Proportion of this Solid, when the points Q and P coincide.
For



Fig. 40

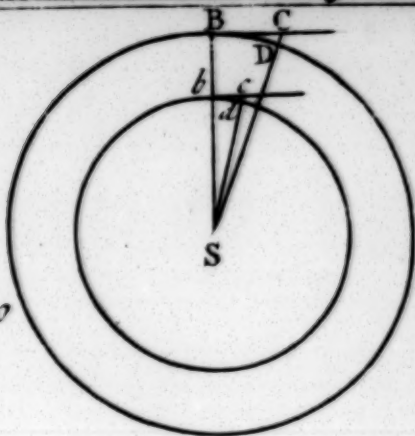


Fig. 41

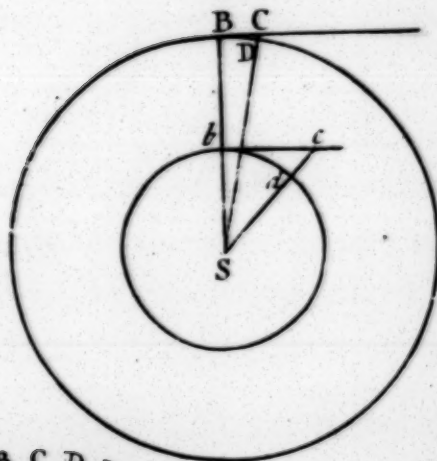


Fig. 42

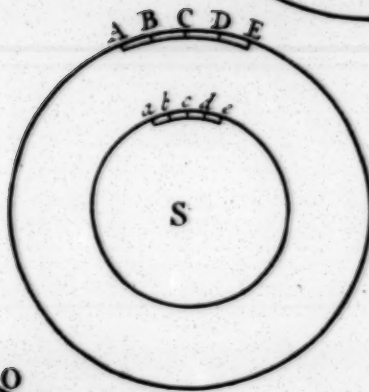
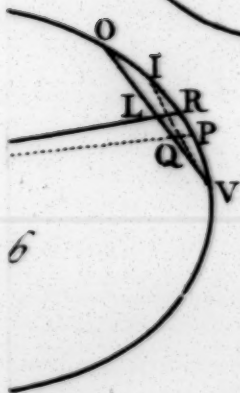


Fig. 44



6

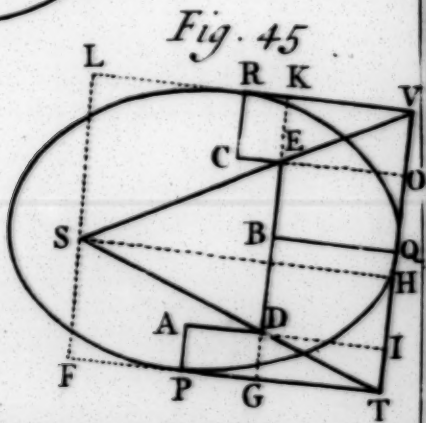


Fig. 45

2

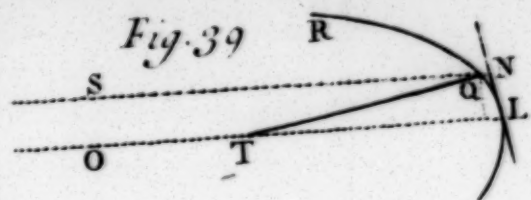


Fig. 39

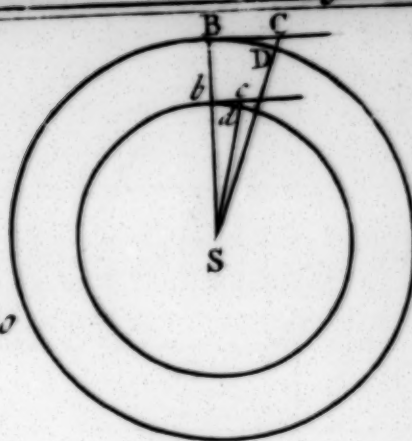


Fig. 40

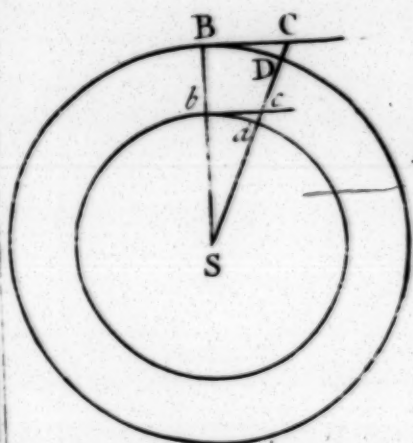


Fig. 41

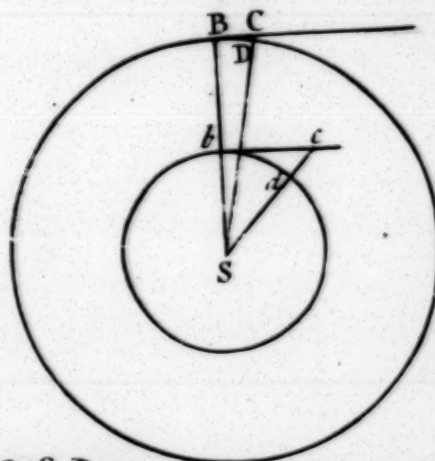


Fig. 42

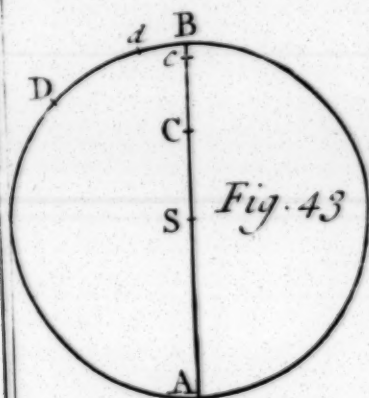


Fig. 43

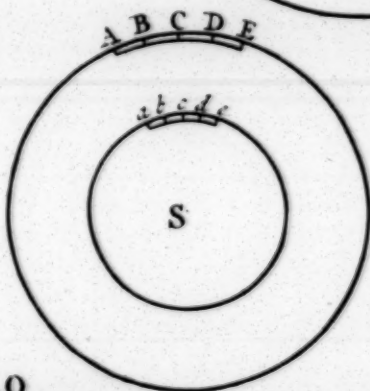


Fig. 44

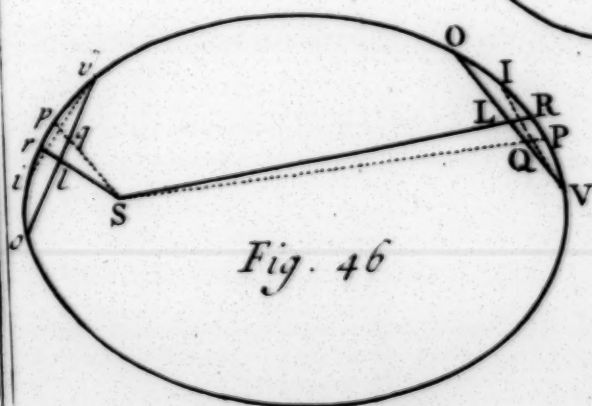


Fig. 46

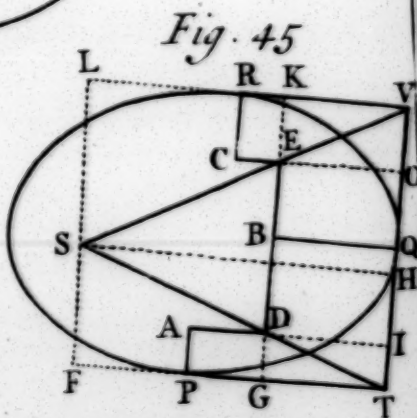


Fig. 45

2

For QR is equal to the versed Sine of twice the Arch QP , in the middle of which the point P is, (*by the Construction*;) and twice the Triangle SQP , which is equal to $SP \times QT$, (*by the Elements*;) is proportionable to the Time in which twice the Arch QP is described: (*by Prop. I.*) wherefore if in the Proportion above, *viz.* $\frac{LR}{T^2} = C$, we put QR , instead of LR , and $SP^2 \times QT^2$, instead of T^2 , it will be $C = \frac{I}{\frac{SP^2 \times QT^2}{QR}}$, that is, the centripetal Force is as $\frac{SP^2 \times QT^2}{QR}$ reciprocally.

Corol. 2. By the same Argument the centripetal Force is reciprocally as the Solid $\frac{SY^2 \times QP^2}{QR}$, if SY be a Perpendicular let fall from S , the Center of the Forces, to the Tangent of the Orb PR . For the point Q being infinitely near R , the right-angled Triangles QTP , SYP , are similar (*by the Elements.*) and consequently the Rectangles $SY \times QP$, and $SP \times QT$, are equal; therefore (*by Substitution in the Proportion of the foregoing Corollary*) $\frac{SP^2 \times QT^2}{QR}$ is equal to $\frac{SY^2 \times QP^2}{QR}$.

Concerning CENTRIPETAL FORCES.

Corol. 3. IF the Orb APQ, be a Circle, or touches a Circle concentrically, or cuts it concentrically, that is, if it makes QPV the Angle of Contact or of Section with the Circle, the least that can be, having the same Curvature, and the same Radius of Curvature at the point P, viz. CP. (*See Milnes's Conick Sections, Part III. Prop. X.*) And if PV be a Chord of this Circle drawn from the Body through the Center of Forces; the centripetal Force will be reciprocally as the Solid $SY^2 \times PV$. For (*by the Property of the Circle*) $VT \times TP = QT^2$, when Q is infinitely near P; and TP being infinitely less, or bearing no finite Proportion to VT; VP and VT may be looked upon as equal, or may be put for each other (*by Lemma I.*)

Wherefore $PV = \frac{QP^2}{QR}$. Let PV be put in the Proportion of the foregoing Corollary $SY^2 \times \frac{PQ^2}{QR}$ for $\frac{PQ^2}{QR}$; and it will be $SY^2 \times PV$.

Corol. 4. THE same things being supposed, the centripetal Force is as the Square of the Velocity directly, and the Chord PV inversely. For the Velocity is reciprocally as the Perpendicular SY (*by Cor. 1. Prop. I.*)

or $V^2 = \frac{I}{SY^2}$: Therefore, if in the Proportion of the foregoing Corollary $SY^2 \times PV$, for SY^2 , we put $\frac{I}{V^2}$; the Reciprocal of $SY^2 \times PV$,

$SY^2 \times V$, will become $\frac{V^2}{PV}$, that is, as the Square of the Velocity directly, and the Chord PV inversely.

Corol. 5. HENCE it follows; that if any curvilinear Figure APQ be given, and there be also given any point S in it, to which the centripetal Force is continually directed; we may find the Law of the centripetal Force by which any Body P is perpetually drawn from its rectilinear Course, and kept in the Perimeter of that Figure, and by revolving describes it. For this Force is every where

as the Solid $\frac{SP^2 \times QT^2}{QR}$, or as the Solid

$SY^2 \times PV$; now these are general Proportions, and will hold for all Curves; therefore when any particular Curve is given, as a Circle, Ellipsis, Spiral, &c. and the particular Point to which the Forces are directed, is also given, as the Center, Focus, or any other Point; we must find out the Value or Proportion of these general Solids in the known Properties of the given Curves, so will the Law of the Forces be given in known Terms. Of this we shall give Examples in the following Problems.

PROPOSITION VII. PROBLEM II.

If a Body revolve in the Circumference of a Circle; it is required to find the Law
L 4
of

Concerning CENTRIPETAL FORCES.

of the centripetal Force, tending to any given Point whatsoever.

Fig. 48. LET VQPA be the Circumference of a Circle; S the given Point to which the Force tends as to its Center. P the Body revolving in the Circumference; Q the place into which it immediately moves out of P. and PRZ a Tangent of the Circle at the point P. Let the Chord PV be drawn through the point S, and having drawn the Diameter of the Circle VA, let A, P be joined; further, from the point Q let QT be let fall perpendicular to SP, and produced till it meets the Tangent PR in Z; lastly, through the point Q let LR be drawn parallel to SP, meeting the Circle in L, and the Tangent PZ in R. Now because QR is parallel to TP, and ZT parallel to PA, the Triangles ZQR, ZTP, and VPA, are similar (*by the Elements*;) and therefore RP^2 that is, QRL (*by the Property of the Circle and Lemma I.*) is to QT^2 , as AV^2 to PV^2 . consequently $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2 . Let these two equal Quantities be multiplied by $\frac{SP^2}{QR}$, and let PV be put instead of RL, when the point Q coincides with P, (*by Lemma I.*) and then $\frac{SP^2 \times PV^3}{AV^2}$ will

will be equal to $\frac{SP^2 \times QT^2}{QR}$. But (by Cor. 1. and 5th of Prop. VI.) the centripetal Force is reciprocally as $\frac{SP^2 \times QT^2}{QR}$; therefore it is also reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$, which is equal to $\frac{SP^2 \times QT^2}{QR}$; that is, (because AV^2 is given, or is an unalterable Quantity, and therefore makes no Variation in the Proportion, and consequently may be left out) it is reciprocally as $SP^2 \times PV^3$, or reciprocally as the Square of the Distance or Height SP , and the Cube of the Chord PV together. W. W. S.

The same demonstrated another way.

LET the Tangent PR be produced, and SY be let fall perpendicular to it, then the Triangle ZTP being similar to each of the Triangles SYP and VPA (because the Angle SPY is common to both the Triangles SYP, ZTP , and is also equal to the Angle VAP , by Prop. 32. Book III. of the Elements.) it is manifest, that the Triangles SYP and VPA are similar also; consequently AV , is to PV ; as SP to SY ; and (by Multiplication and Division) $\frac{SP \times PV}{AV}$ is equal to SY . and (by squaring each of these Quantities, and multi-

multiplying them by PV ,) $\frac{SP^2 \times PV^3}{AV^2}$ will be equal to $SY^2 \times PV$. And therefore (by Substitution, as before in Corol. 3^d and 5th, Prop. VI.) the centripetal Force will be reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$, that is, reciprocally as $SP^2 \times PV^3$ (because AV^2 is a given Quantity.) W. W. S.

Cor. 1. HENCE it follows, that if the given point S , to which the centripetal Force always tends, be placed any where in the Circumference of the Circle, suppose at V ; then the centripetal Force will be reciprocally as the quadrato Cube, or fifth Power of the Distance or Altitude SP ; because in this Case SP^2 becomes equal to PV^2 , consequently, if in the Quantity $SP^2 \times PV^3$, we put PV^2 for its equal PS^2 , it will be $SP^2 \times PV^3 = PV^5$. W. W. S.

Cor. 2. THE Force with which a Body
 Fig. 49. P is revolved in the Circle $APT V$, is to the Force with which the same Body P may be revolved in the same Circle, about any other Center of Forces R , and in the same periodical Time (for if the periodical Time be not the same, the Velocity of the revolving Body P will be increased or decreased, and consequently the centripetal Force will also be increased or decreased, as the Square of the Velocity by Lem. XI.) as $RP^2 \times SP$ to the Cube of the straight Line SG , drawn from the first Center of Forces, to the Tangent of the Orb PG , and parallel to RP , the Distance of the
 same

same Body from the other Center of Forces. For by the Construction of this Proposition, the first Force towards the Center R, is to the second Force towards the Center S; as $RP^2 \times PT^3$, to $SP^2 \times PV^3$, or (*multiplying each of these Quantities by SP, and transposing PT³*) the same Forces will be to each other, as $RP^2 \times SP$ to $\frac{SP^3 \times PV^3}{PT^3}$. Now be-

cause RP is parallel to SG, the Triangles PSG and TPV are similar, and consequently $SG:SP::PV:PT$. and (*by multiplying the Extremes and Means, and transposing PT*;) it will be $SG = \frac{SP \times PV}{PT}$, and

by Involution, $SG^3 = \frac{SP^3 \times PV^3}{PT^3}$. Let SG^3

be substituted for $\frac{SP^3 \times PV^3}{PT^3}$, and the centripetal Forces will be to each other, as $RP^2 \times SP$, to SG^3 .

Cor. 3. THE Force with which the Body P is revolved about the Center S in any Orb whatsoever (*Ellipsis, Oval, &c.*) is to the Force with which the same Body P may be revolved about any other Center of the Forces R, and in the same periodical Time; as $SP \times RP^2$, that is, as the Distance of the Body from the first Center of Forces S, multiplied by the Square of its Distance from the other Center of Forces R; to the Cube of the streight Line SG, drawn from the first

Fig. 50.

first Center of Forces S to the Tangent of the Orb PG ; and parallel to RP , the Distance of the same Body from the second Center of Forces. For the Forces in this Orb are the same in any point P whatsoever, as they are in a Circle of the same Curvature. For let PC be drawn perpendicular to PG , and equal to the Radius of Curvature; and on the Center C , with this distance, let the Circle BPO be described, which will be equicurval to the point P of the *Ellipsis* (by *Prop. X, &c. Part III. of Milnes's Conick Sections*) consequently the centripetal Force in the point P , is the same, whether the Body P moves in the *Ellipsis* $APT V$, or in the Circle BPO ; because at their Interfection or Coincidence P , the elementary Arches or their versed Sines are equal; and therefore the Proportion of this same Force when tending to S or to R , is the same as before, viz. as $RP^2 \times SP$ to SG^3 .

PROPOSITION VIII. PROBLEM III.

Let a Body be moved in a Semicircle PQA : In order hereunto it is necessary that the centripetal Force should tend to a Point S , at so great a Distance, that all the Lines PS, RS , drawn to that Point, may be looked upon as parallel; otherwise the Body will move in a whole Circle about S , as its Center: upon this Supposition, it is required to find the Law of the centripetal Force tending to S , at an infinite Distance.

FROM

FROM C, the Center of the Semicircle, let the Semidiameter CA be drawn, cutting the Fig. 51. parallel Lines PR, RS, in M and N, and let CP be joined. Then because QR is parallel to PT, the Triangles PZT, and RZQ are similar; and because the Angle ZPT, is the Complement of the Angles MPC, and PZT, to a right Angle, the Triangles MPC, and ZPT, are also similar; consequently (by squaring the homologous or corresponding Sides) CP^2 , is to PM^2 ; as PR^2 , to QT^2 ; and (by the Property of the Circle) PR^2 is equal to the Rectangle $QR \times RN \times QN$; that is, when the Points Q and P coincide (by Lemma I.) PR^2 is equal to the Rectangle $QR \times 2PM$. Therefore (by Substitution) CP^2 , is to PM^2 ; as $QR \times 2PM$, to QT^2 , and (by Multiplication and Transposition) $\frac{QT^2}{QR}$, is equal to $\frac{2PM^3}{CP^2}$. And again (multiplying each of these Quantities by SP^2 ;) it will be $\frac{QT^2 \times SP^2}{QR}$, equal to $\frac{2PM^3 \times SP^2}{CP^2}$. Wherefore (by Substitution in the Proportion of Cor. 1. and 5. Prop. VI.) the centripetal Force is reciprocally as $\frac{2PM^3 \times SP^2}{CP^2}$; that is, leaving out the determinate or given Proportion $\frac{2SP^2}{CP^2}$, the centripetal Force is reciprocally as PM^3 .
W. W. D.

THE

THE same Proportion may easily be collected from the foregoing Proposition. For the centripetal Force was there reciprocally as $SP^2 \times PV^3$; but when SP becomes infinite, it may be left out of the Proportion, and PV will then become proportionable to MP , because MS is infinite also; so that in this Case PV^3 will be proportionable to PM^3 , and may be put for it. W. W. D.

S C O L I U M.

BY pretty much the same way of Arguing, a Body will be found to move in an Ellipsis, or an Hyperbola, or a Parabola, with a centripetal Force, which is reciprocally as the Cube of the ordinate Applicate tending to the Center of Forces supposed to be at the greatest Distance possible; for in every one of these Figures PR^2 is proportionable to the Rectangle $QP \times 2PM$, by the Properties of the Conick Sections. (See *L'Hospital, Book IV. Prop. XIII.*)

PROPOSITION IX. PROBLEM IV.

Let a Body revolve in the Spiral PQ, pqS , cutting all the Radius's SP, SQ , &c. in a given Angle, that is, so that the Angles QPS, pQS , &c. may be equal; it is requir'd to find the Law of the centripetal Force tending to S , the Center of the Spiral.

LET

LET the indefinitely small Angles PSQ , pSq , be given, or be equal to each other, then Fig. 52. because the Angles QPS , QSP ; and qpS , qSp , are equal to each other; (*by the Hypothesis*) therefore the Triangles SQP , Sqp , are similar (*by the Elements.*) and because QR is parallel and QT perpendicular to SP , and qr parallel, and qt perpendicular to Sp ; therefore the Figures $SPRQT$, and $Sprqt$, are given in Species, that is, are of the same Kind, or similar to each other, and consequently all the homologous or correspondent Sides of these Figures are proportionable to the whole, and to each other (*by the Elements.*) Therefore the Proportion of QT to QR is given, that is, $QT, QR :: qt : qr$. and $\frac{QT^2}{QR}$, is as QT , that is (*because the Species or*

Kind of the Figure is given) as SP ; or $\frac{QT^2}{QR}$

: $SP :: \frac{qt^2}{qr} : Sp$. Let the Angles PSQ , pSq ,

be any way alter'd by increasing or diminishing them; then (*by Lemma XI.*) the streight Lines QR , qr , subtending the Angles of Contact QPR , qpr , will be altered in a duplicate Proportion of PR or QT ; pr or qt , that is, $QR : qr :: PR^2$ or $QT^2 : pr^2$ or qt^2 . Therefore however the Angles PSQ , pSq , be altered, the Proportion of $\frac{QT^2}{QR}$ to SP will re-

main

main the same as before, viz. as $\frac{q t^2}{q r}$ to $S p$.

Wherefore if each of these Proportions be multiplied SP^2 , Sp^2 , it will be $\frac{QT^2 \times SP^2}{QR} : SP^3 ::$

$\frac{q t^2 \times Sp^2}{q r} : Sp^3$. Therefore (by *Corol. 1. and*

5. Prop. 6.) the centripetal Force is reciprocally as the Cube of the Distance SP . W. W. D.

The same demonstrated another way.

LET SY be let fall from S perpendicular
 Fig. 53. to the Tangent PY , then let PC be drawn perpendicular to PY , or parallel to SY , and LC perpendicular to SP , the Intersection C of these two Lines will be the Radius of the Circle PV , equicurval to the point of the Spiral P . (See *L'Hospital Infiniment petits. Section V. § 91.*) and PV , the Chord of the Circle, cutting the Spiral concentrically. Now because the Triangle SPY is given, to which the Triangle LPC is similar, CP being parallel to SY ; and because CP is the Radius of the Circle PV , therefore the Proportions of the Perpendicular SY , and the Chord PV to the Distance SP are given; that is, $SY \times PV$ is proportionable to SP , and consequently any Composition of these Quantities are proportionable to each other: Therefore SP^3 is proportionable to $SY^2 \times PV$, that is, (by *Corol. 3.*
and

and 5. Prop. VI.) SP^3 is as the centripetal Force reciprocally.

SCHOLIUM.

BUT the contrary to this does not follow, viz. that if the centripetal Force be reciprocally as the Cube of the Distance, then the Body must revolve in a Spiral, for (by Cor. 8. Prop. IV.) it may be made to revolve in a Circle.

LEMMA XII.

All the Parallelograms described about any conjugate Diameters of a given Ellipsis or Hyperbola, are equal to each other.

THIS Lemma is demonstrated in the XIth Proposition of the second Part of Milnes's Conick Sections.

PROPOSITION X. PROBLEM V.

Let a Body be revolved in an Ellipsis, it is required to find the Law of the centripetal Force tending to the Center of the Ellipsis.

LET CA, CB, be the Semiaxes of the Ellipsis; GP, DK, two other conjugate Diameters; PF, QT, Perpendiculars to these Diameters; Qv, an ordinate Applicate to the
M Diame-

Fig. 54.

Diameter GP. Now if the Parallelogram $QvPR$ be compleated, then (*by Prop. XVIII. Part I. of Milnes's Conick Sections.*)

$Pv^2 \times vG : Qv^2 :: PC^2 : CD^2$. But because Qv is parallel to CK , therefore the Angles Q , vT and PCF are equal, and consequently the Triangles QTv and PCF are similar: whence

$Qv^2 : QT^2 :: PC^2 : PF^2$. and (*multiplying these Proportions into each other, and dividing by Qv^2 .*)

$Pv \times vG : QT^2 :: PC^2 \times PC^2 : CD^2 \times PF^2$ and (*by Transposition.*)

$$vG : \frac{QT^2}{Pv} :: PC^2 : \frac{CD^2 \times PF^2}{PC^2}. \text{ Let}$$

QR be put for Pv , and (*by Lemma XII.*) $BC \times CA$ be put for $CD \times PF$, and when the points Q and P coincide, let $2PC$ be put for vG (*by Lemma I.*) and let the middle Terms and the extreme Terms be multiplied together; and it will be $\frac{QT^2 \times PC^2}{QR} = \frac{2BC^2 \times CA^2}{PC}$.

Wherefore (*by Cor. 5. Prop. VI.*) the centripetal

petal Force will be reciprocally as $\frac{2BC^2 \times CA^2}{PC}$; that is, (because $2BC^2 \times CA^2$ is a given Quantity, and may be left out of the Proportion) reciprocally as $\frac{1}{PC}$, or directly as the Distance PC. W. W. D.

The same demonstrated another way.

IN the streight Line PG on the other side of T, let the point u be taken, so that Tu may be equal to Tv ; and let uV be so taken, that it may be $uV : vG :: DC^2 : PC^2$. Now, because (by Prop. XVIII. Part I. of Milnes's Conick Sections) $Qv^2 : Pv \times vG :: DC^2 : PC^2$. Therefore (by the Elements) $uV : vG :: Qv^2 : Pv \times vG$. And multiplying the Extremes and Means, and dividing by vG , it will be $Qv^2 = Pv \times uV$. Let the Rectangle $uP \times Pv$ be added to each side of this Equation, and the Square of the Chord of the Arch QP will become equal to the Rectangle $Vp \times Pv$. For (by the Elements.) $QP^2 = QT^2 + TP^2$, and $QT^2 = Qv^2 - Tv^2$. Therefore (by Substitution) $QP^2 = Qv^2 + TP^2 - Tv^2$: But $TP^2 - Tv^2 = TP + Tv$ (or Tu) = uP multiplied by $TP - Tv = Pv$. Consequently $QP^2 = Qv^2 + uP \times Pv$. And $Pv \times uV + uP \times Pv = uV + uP \times Pv = Vp \times Pv$. Wherefore (by the Property of the Circle.) a Circle which touches the Ellipsis in P, and passes through the point

M 2 Q will

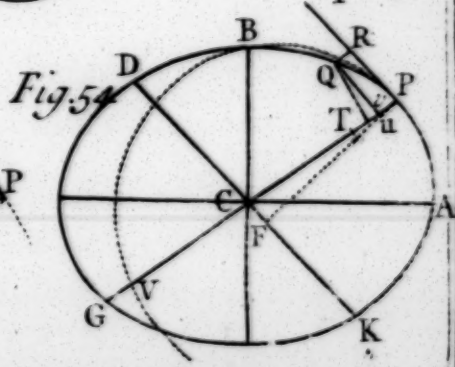
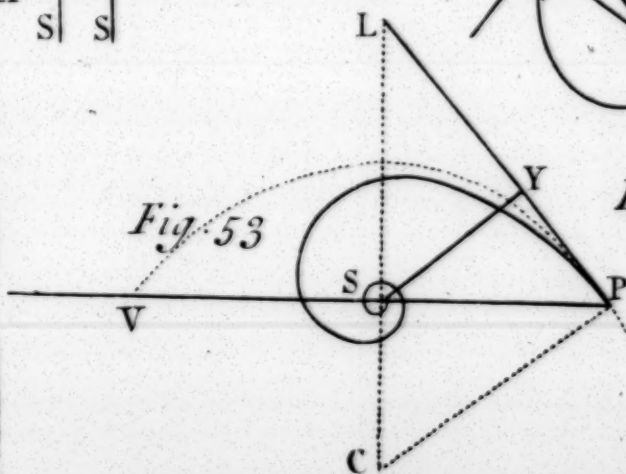
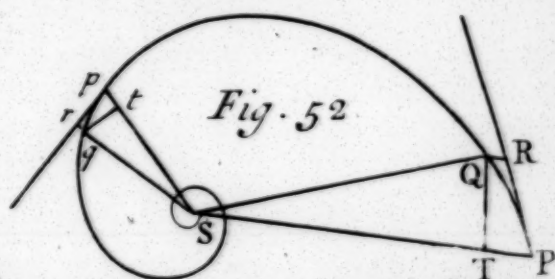
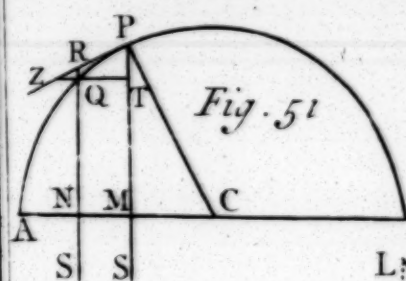
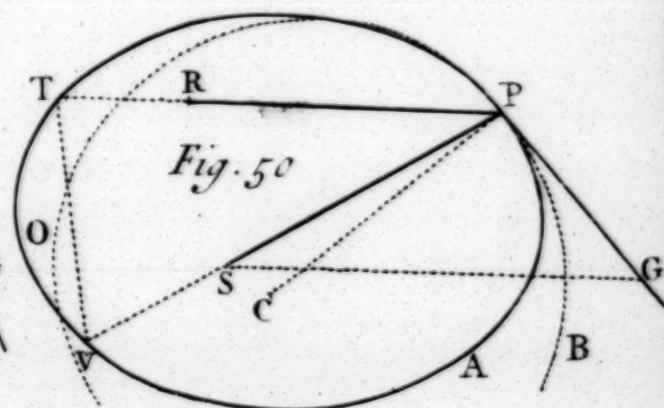
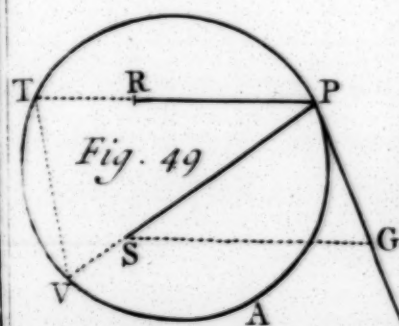
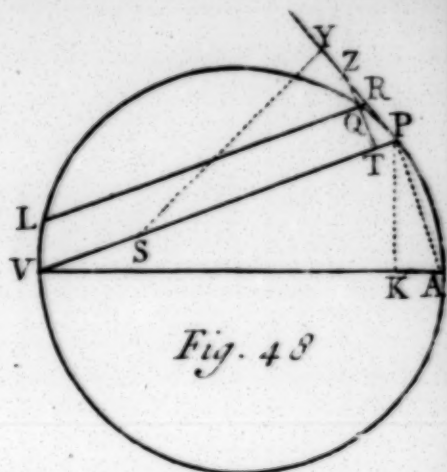
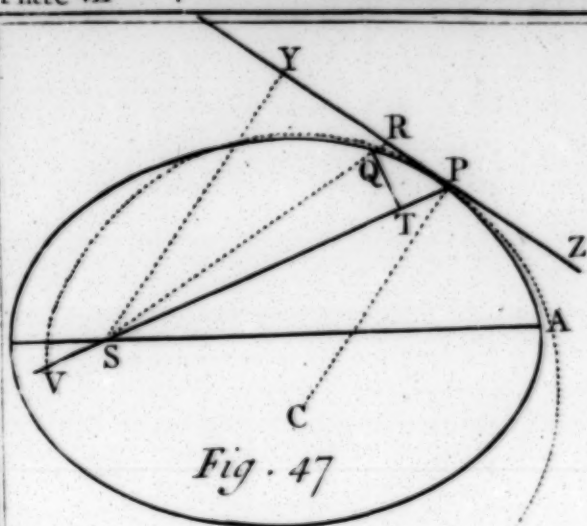
Q, will also pass through the point V. Then suppose the points P and Q to coincide, and the Proportion of uV to vG , (which is the same as the Proportion of DC^2 to PC^2 ,) will become the Proportion of PV to PG , or of PV to $2PC$. Therefore $DC^2 : PC^2 :: PV : 2PC$. And (by Multiplication and Division.) $PV = \frac{2DC^2}{PC}$. Whence the Force

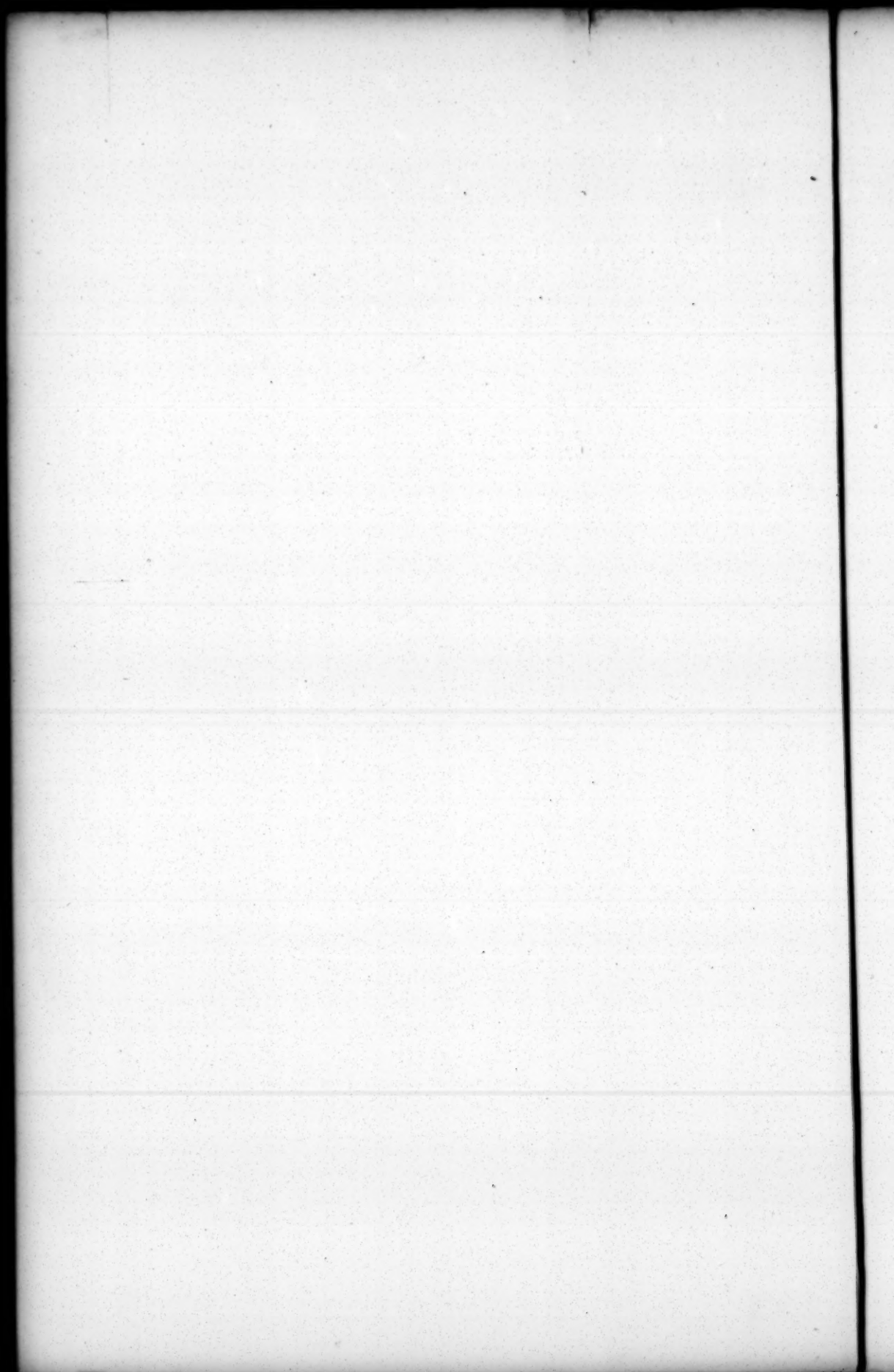
with which the Body P is revolved in the Ellipsis, will be as $\frac{2DC^2}{PC} \times PF^2$ reciprocally (by

Corol. 3. Prop. VI.) that is, as PC directly, (for $2DC^2$.) $\times PF^2$ is a given Quantity, and makes no Alteration in the Proportion.) W. W. D.

Corol. 1. The centripetal Force therefore is as the Distance of the Body from the Center of the Ellipsis. And on the other hand, if the centripetal Force be as the Distance, a Body will be moved in an Ellipsis, whose Center is the Center of the Forces, or else in a Circle into which such an Ellipsis may easily be changed (by *Corol. 3. Prop. IV.*)

Corol. 2. And the periodical Times of the Revolutions, are the same in all Ellipses whatsoever which are described about the same Center. In similar Ellipses the periodical Times are equal; for if in different concentrick Circles the centripetal Forces be as the Radius's, the Velocities will be also as the Radius's, and the periodical Times will be equal; (by *Corol. 3. Prop. IV.*) and if (by *Corol. 10. of the same Prop.*)





Prop.) instead of the Radius's of such Circles, we substitute the Distances in similar Ellipses, the Demonstration will hold the same. Thus let BEA, bea , be two similar Ellipses described about their common Center S ; if the centripetal Forces in the same Ellipsis, be as the Distances from the Center, the Velocities will be as those Distances also, that is $Sb : Se :: bd : eg$. consequently the triangular Spaces Sbd, Seg , are similar, and the Times in which db and eg are described are equal: For the same Reason, the Times in which db and DB in different Ellipses are described, are equal also. Therefore (*by Composition.*) as the small Particle of Time in which bd is described, is to the small Particle of Time in which BD is described; so is the periodical Time in which the whole Ellipsis bea is described, to the periodical Time in which the whole Ellipsis BEA is described; (*each of them being composed of an equal Number of similar Areas Sbd, SBD .*) but the former of these Proportions are equal, and therefore so are the latter. If the Ellipses be not similar, but have the same common longer Axis, then the periodical Times will be as the whole Areas of the Ellipses directly, and the Particles of the Areas described in the same Time inversely. Thus suppose AEC, AFC , to be two concentrick Ellipses described upon the same longer Axis CA , whose periodical Times are P, Π ; and let the Particles SBC, ScC of those Areas described in the same Moment of Time be a, α ; the Areas themselves A, A ; and the

Fig. 55.

Fig. 56.

given Moment of Time t . Then it will be

$$P : t :: A : a.$$

and $t : \Pi :: a : A$ (by *Prop. I.*) and by

$$P : \Pi :: A : A \text{ Composition and Division. } \frac{A}{a} : \frac{A}{a}.$$

But the whole Areas A, A , are to each other as the lesser Semiaxes ES, FS , (by *Prop. VI. Book V. of L'Hospital's Conick Sections.*) and the Particles of the Areas a, a , are to each other as the Velocities in their principal Vertexes, or as the small Lines Bc, cD , described in the same time; because the Triangles BSD, cSD being of the same Altitude DS , their Areas are to each other as their Bases, BD, cD ; (by *the Elements.*) let u, v , be put for these Velocities, then by Substitution in the Proportion $P : \Pi :: \frac{A}{a} : \frac{A}{a}$, it will be-

$$\text{come } P : \Pi :: \frac{ES}{u} : \frac{FS}{v}, \text{ that is, the periodical}$$

Times are as the lesser Semiaxes directly, and the Velocities in the principal Vertexes inversely. But (by *the foremention'd Prop. of L'Hospital.*) $u : v$, or $BD : cD :: ES : FS$. Therefore if for u, v , we put ES, FS ; it will be,

$$P : \Pi :: \frac{ES}{ES} : \frac{FS}{FS}, \text{ that is, the periodical Times}$$

are as the lesser Semiaxes ES, FS , directly, and BD, CD , the Ordinates applicate to the same point of the common Axis inversely; that is, (because these direct and inverse Proportions

tions are equal, or $EF \times FS = EF \times FS$) the periodical Times are equal.

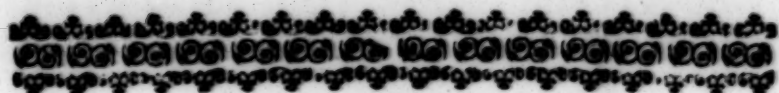
Now similar Ellipses, and Ellipses described upon the same Axis, take in all possible Variety of Ellipses; for an Ellipsis similar to any particular Ellipsis may be described upon the Axis of any other Ellipsis; consequently, the periodical Times in all Ellipses whatsoever, is the same. Thus on the Axis ac , let the Ellipsis *abcd* be described similar to the Ellipsis $ABCD$; then the periodical Time in the Ellipsis $ABCD$, will be equal to the periodical Time in the Ellipsis *abcd*, (*because these Ellipses are similar by the Hypothesis:*) Then if the Ellipsis $aEcF$ be described on the Axis ac , the periodical Time in the Ellipsis *abcd*, will be equal to the periodical Time in the Ellipsis $aEcF$ (*because these Ellipses are both described on the same common Axis ab*) therefore the periodical Time in the first Ellipsis $ABCD$, is equal to the periodical Time in the last Ellipsis $aEcF$ (*by Equality of Proportion.*) Fig. 56.

SCHOLIUM.

If the Center of the Ellipsis go off infinitely, so that it be turned into a Parabola, then the Forces tending to the Center at an infinite Distance will become equable, because all the infinite Distances from the several points of the Parabola are equal (*by Lemma I.*) And this is Galilæus's Theorem. And if the Parabolick

Section of the Cone (by altering the Inclination of the Plane to the Cone) be turned into an Hyperbola, the Body will be moved in its Perimeter, and the centripetal Force will be turned into a centrifugal Force. And as in a

Fig. 57. Circle, or in an Ellipsis ABCD, if the centripetal Forces tend to the Center S, placed in the Absciss of the Figure FB produced; these Forces, by increasing or diminishing the Ordinates EF, AS, in any given Proportion whatsoever, or else by altering the Angle of the Inclination of the Ordinates to the Absciss, *viz.* the Angle EFB; the Forces, I say, will always be increased or diminished in proportion to the Distances from the Center, provided the periodical Times continue equal: So likewise in all sorts of Figures whatsoever, the Proportions of whose Abscisses to the Ordinates are expressed by any Equation, if the Ordinates be increased or decreased in any given Proportion whatsoever, or the Angle of Ordination EFB be altered any way whatsoever; so long as the periodical Times continue equal, the Forces tending to any Center S, placed in the Absciss BF, will be increased or decreased in every one of the Ordinates EF, in proportion to their Distances from the Center S.



SECTION III.

Concerning the Motion of Bodies in Conick Sections, where the Forces do not tend to the Centers of the Sections.

PROPOSITION XI. PROBLEM VI.

Let a Body revolve in an Ellipsis: it is required to find the Law of the centripetal Force tending to the Focus of the Ellipsis.

LET S be the Focus of the Ellipsis. Let the Fig. 58.
 Line SP be drawn so as to cut the Diameter of the Ellipsis DK in E, and the Ordinate applicate Qv in x, and let the Parallelogram QxPR be compleated. It is evident that EP is equal to the longer Semiaxis AC; for if from H, the other Focus of the Ellipsis, the Line HI be drawn parallel to EC, then because CS and CH are equal, and the Triangles SEC, SIH similar, ES will be equal to EI (by the Elements.) consequently if EI be added to IP, and $ES = EI$ be taken from SP; EP will be equal to half $PS + PI$, that is, $PS + PH$; (because HI is parallel to PR, and the Angles IPR

Concerning CENTRIPETAL FORCES.

IPR, HPZ, are equal, (by *Prop. III. Part IV. of Milnes's Conick Sections*;) but $SP + PH$ is equal to $2CA$ (by *Prop. V. Part IV. of Milnes's Conick Sections*;) therefore EP is equal to CA. Let QT be let fall perpendicular to SP, and let the principal *Latus Rectum* of the Ellipsis (or $\frac{2BC^2}{AC}$ by *Prop. XXIV. Part I. of Milnes's Conick Sections*) be called L; then it will be.

$$L \times QR : L \times Pv :: QR : Pv; \text{ (by equal Multiplication,)}$$

that is, :: PE or AC : PC; because

$QR = xP$,
and the Triangles xPv
and EPC ,
are similar)

$$\text{And } L \times Pv : Gv \times Pv :: L : Gv; \text{ (by equal Multiplication;)}$$

$$\text{And } Gv \times Pv : Qv^2 :: PC^2 : CD^2 \text{ (by Prop. XVIII.}$$

Part I. of Milnes's Conick Sections.

And when the Points Q and P coincide, $Qv^2 = Qx^2$

$= Qx^2$ (by
Cor. 2.

Lem. VII.)

And Qx^2 or $Qv^2 : QT^2 :: EP^2 : PF^2$. For Qx
being pa-
rallel to
EF, the
Angles
FEP and
 QxT are
equal, and
confe-
quently,
the Trian-
gles EFP,
 xQT , are
fimilar (by
the Ele-
ments.)

that is $:: CA^2 : PF^2$.

or $:: CD^2 : CB^2$. (by Lemma
XII.

Let all these Proportions be multiplied toge-
ther, and the common Quantities in each of
them be struck out, and it will be

$$L \times QR : QT^2 :: AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2;$$

$$\text{or } :: 2CB^2 \times PC \times CD^2 : PC \times Gv \times CD^2 \times CB^2,$$

because $AC \times L = 2CB^2$, as be-
(fore,

Or $L \times QR : QT^2 :: 2PC : Gv$. (by striking
out the equal Quantities in these last Proportions.)

Now when the points Q and P coincide, $2PC$
and Gv will be equal (by Lemma I.) There-
fore

fore $L \times QR$ and QT^2 , which are proportionable to them, will be equal also. Let these two equal Quantities be multiplied by $\frac{SP^2}{QR}$, and it will be $L \times SP^2 = \frac{SP^2 \times QT^2}{QR}$; wherefore (by *Corol. 1. and 5. Prop. VI.*) the centripetal Force is reciprocally as $L \times SP^2$, that is, reciprocally in a duplicate Proportion of the Distance SP , (because the *Latus rectum* L is a given Quantity.) W. W. S.

The same demonstrated another way.

Fig. 58. BECAUSE the Force with which the Body P may be revolved in an Ellipsis, if it tend to the Center, is as cP the Distance of the Body from C , the Center of the Ellipsis; let CE be drawn parallel to PR , the Tangent of the Ellipsis; then if CE and PS meet each other in E , and C be put for the Force with which the Body P is revolved about the Center C , and c be put for the Force with which it is revolved about the Focus S ; then (by *Corol. 3. Prop. VII.*) $C : c :: CP \times SP^2 : EP^3$ (for $SP \times RP^2 : SG^3$ in that *Corol.* is the same as $CP \times SP^2 : EP^3$ here.) but C is proportionable to CP (by the Hypothesis;) therefore $CP : c :: CP \times SP^2 : EP^3$. and (by *Multiplication and Division.*) c , or the Force by which the Body is revolved about the Focus S , is as $\frac{EP^3}{SP^2}$, that is, as SP^2 reciprocally.

reciprocally, because the point S being the Focus of the Ellipsis, PE is a given Quantity, and is equal to CA, as was shewn before. W. W. S.

I MIGHT apply this Problem to the Parabola and Hyperbola, with the same Brevity as was done in Problem the Fifth; but because of the Value of it and its Usefulness in what follows, I shall not think it too troublesome to confirm those other Cases by a Demonstration.

PROPOSITION XII. PROBLEM VII.

Suppose a Body to move in a Hyperbola, it is required to find the Law of the centripetal Force tending to the Focus of this Figure.

LET CA, CB, be the Semiaxes of the Hyperbola, PG, KD, two other conjugate Diameters; and Qv and Ordinate applicate to the Diameter GP. Let SP be drawn so as to cut the Diameter DK in E, and the Ordinate applicate Qv in x, and let the Parallelogram QRPx be compleated. It is manifest, that EP is equal to the transverse Semi-axis AC: for if from H, the other Focus of the Hyperbola, there be drawn the Line HI, parallel to EC, then because CS is equal to CH, and the Triangles CES, HIS, similar (*by the Elements.*) ES will be equal to EI; and if PE be subducted from ES, and added to EI, $2EP + PS$ will be equal to PI, and consequently

frequently EP will be half the Difference between PS and PI, or half the Difference between PS and PH (because IH and PR are parallel (*by Prop. XX. Part I. of Milnes's Conick Sections*), and the Angles IPR, HPZ are equal, (*by Prop. III. Part IV. of Milnes's Conick Sections*) whose whole Difference, that is, PH — PS is equal to the whole Axis 2 AC. Let QT be let fall perpendicular to SP; and let the principal *Latus rectum* of the Hyperbola (or $\frac{2BC^2}{AC}$, *by Prop. XXIV. Part I. of Milnes's Conick Sections*.) be called L; then it will be,

$$L \times QR : L \times Pv :: QR : Pv, \text{ (by equal Multiplication.)}$$

or $Px : Pv$, because RP and Qv are parallel (*by Prop. XX. Part I. of Milnes's Conick Sections*.)

that is, PE : PC. for the Triangles Pxv, PEC are similar, because Qv being parallel to CE, (by

(by Prop.
XX. Part I.
of Milnes's
Conick Sec-
tions,) the
Angles
 Pxv, CEP
and $Pvx,$
 $EC P$ are
equal (by
the Ele-
ments.)

or $AC : PC$, for $AC =$
 PE , as was
proved be-
fore.

Also $L \times Pv : Gv \times Pv :: L : Gv$, (by equal
Multiplica-
tion.)

And $Gv \times vP : Qv^2 :: PC^2 : CD^2$ (by Prop.
XVI.)
Part I. of
Milnes's Co-
nick Sec-
tions.

And Qx^2 or $Qv^2 : QT^2 :: EP^2 : PF^2$. for when
the points
 Q and P co-
incide, Qv^2
and Qx^2 are
equal (by
Cor.
Lem. VII.)
that

that is, $CA^2:PF^2$. for $EP^2 = CA^2$, as before.

or $CD^2:CB^2$ (by *Lemma XII.*)

Let all the correspondent Terms in these Proportions be multiplied into each other, and divided by those that are common, and it will be.

$$L \times QR : QT^2 :: AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2,$$

$$\text{or } L \times QR : QT^2 :: 2CB^2 \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2,$$

because $2CB^2 = AC \times L$ (by *Prop. XXIV. Part I. of Milnes's Conick Sections.*) And by striking out the common Terms in these last Proportions, it will be

$L \times QR : QT^2 :: 2PC : Gv$. But when the points P and Q coincide, $2PC$ will be equal to Gv , and therefore $L \times QR$ and QT^2 , which are proportionable to them, will be equal also. Let these two equal Quantities be multiplied by $\frac{SP^2}{QR}$, and it will be $L \times P^2$ equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by *Corol. 1. and 5. Prop. VI.*) the centripetal Force is reciprocally as $L \times SP^2$, that is, in a duplicate Proportion of the Distance SP reciprocally, because L is a given Quantity. W. W. S.

The same demonstrated another way,

LET the Force tending to the Center C of Fig. 59. the Hyperbola be first found (by *Prop. X. and its*

its Scholium.) and this will appear to be proportionable to the Distance CP. Then because CE is parallel to PR (by Prop. XX. Part I. of Milnes's Conick Sections.) the Tangent of the Hyperbola, and CE and PS meet each other in E; if C be put for the Force with which the Body P is revolved about the Center C, and c be put for the Force with which it is revolved about the Focus S; then (by Cor. 3. Prop. VII.) $C:c::CP \times SP^2:PE^3$ (for $SP \times RP^2:SG^3$ in that Corollary, is the same as $CP \times SP^2:EP^3$ here.) but C is proportionable to CP, as before; therefore $CP:c::CP \times SP^2:EP^3$, and (by Multiplication and Division.) c, or the Force by which the Body is moved in the Perimeter of the Hyperbola, is as $\frac{EP^3}{SP^2}$, that is, reciprocally as SP^2 , because $EP=CA$ is a given or determinate Quantity, as before. W. W. S.

IN the same manner it may be demonstrated, that the Body will be moved in the opposite Hyperbola GH, by this same centripetal Force changed into a centrifugal one.

LEMMA XIII.

In a Parabola AP, the Latus rectum or Parameter belonging to any Vertex whatsoever P, is equal to four times the Distance of that Vertex from the Focus S of the Figure, or $L=4SP$. Fig. 60.

THIS is demonstrated in Part IV. Prop. X. and its Corol. of Milnes's Conick Sections.

N

LEMMA

L E M M A XIV.

A Perpendicular which is let fall from the Focus of a Parabola to its Tangent, is a mean Proportional between the Distance of the Focus from the Point of Contact, and the Distance of the same Focus from the principal Vertex of the Figure, or $PS : SN :: SN : SA$.

Fig. 60.

LET AP be the Parabola, S its Focus, A the principal Vertex, P the point of Contact, PO an Ordinate applicate to the principal Diameter AO; PM a Tangent meeting the principal Diameter in M, and SN a Line let fall from the Focus perpendicular to the Tangent. Let AN be joined, then because MS is equal to SP (*by Prop. IV. Part IV. of Milnes's Conick Sections.*) MN equal to NP (*by the Elements.*) and MA equal to AO (*by Corol. 2. Prop. III. Part II. of Milnes's Conick Sections.*) therefore NA and OP are parallel (*by the Elements,*) and consequently the Triangle SAN is rectangular at A, and similar to the equal right-angled Triangles SNM, SMP (*by the Elements:*) so that $PS : SN :: SN : SA$.
Q. E. D.

Corol. 1. $PS^2 : SN^2 :: PS : SA$, by multiplying the Antecedents and Consequents by PS, SN, and dividing the latter by SN.

Corol. 2. SN^2 is proportionable to PS, because SA is a given Quantity.

Corol.

Coral. 3. AND the point N where any Tangent PM meets with the streight Line SN, let fall perpendicular to it from the Focus; is in the right Line AN, which touches the Parabola in the principal Vertex.

PROPOSITION XIII. PROBLEM VIII.

Let a Body be moved in the Perimeter of a Parabola; it is required to find the Law of the centripetal Force tending to the Focus of this Figure.

LET the Construction of the Lemma be continued, and let P be the Body in the Peri- Fig. 61.
meter of the Parabola, and from the point Q, the next place that the Body is moved into, let QR be drawn parallel and QT perpendicular to SP; also let Qv be drawn parallel to the Tangent meeting the Diameter PG in v, and the Distance PS in x. Now Px or QR is equal to Pv; for PG being parallel to MS, the Angles MSP, xPv are equal, and the vertical Angles Pxv and QxT or MPS (Qx being parallel to RP by the Construction.) are also equal, therefore the Triangles MSP, xPv are similar (by the Elements.) and consequently MS (as was shewn in the Lemma.) being equal to SP, Px will be equal to Pv. But (by Prop. XXV. Part I. of Milnes's Conick Sections.) the Square of the Ordinate Qv is equal to the Rectangle made up of the

N 2 Latus

Latus rectum, and the Segment of the Diameter Pv , or $Qv^2 = L \times Pv$, that is, (*because* L , by *Lemma XIII.* is equal to $4PS$.) $Qv^2 = 4PS \times Pv$, or $4PS \times QR$; and when the points P and Q coincide, the Proportion of Qv to Qx will become equal (*by Cor. 2. Lem. VII.*) wherefore in this Case Qx^2 is equal to the Rectangle $4PS \times QR$. Then because the Triangles QxT , SPN , are similar (for they are right-angled at T and N , and the Angles x and P are equal, because Qx is parallel to RP .) $Qx^2 : QT^2 :: PS^2 : SN^2$,

that is, $:: PS : SA$ (*by Cor. 1. Lem. XIV.*)

or $:: 4PS \times QR : 4SA \times QR$,
(*by multiplying by $4QR$.*)

wherefore $Qx^2 : QT^2 :: 4PS \times QR : 4SA \times QR$. And because Qx^2 is equal to $4PS \times QR$; QT^2 is equal to $4SA \times QR$ (*by Prop. IX. Book V. of the Elements.*) Let each of these

equal Quantities be multiplied by $\frac{SP^2}{QR}$; and

it will be $\frac{SP^2 \times QT^2}{QR} = SP^2 \times 4SA$, and there-

fore (*by Corol. 1. and 5. Prop. VI.*) the centripetal Force is reciprocally as $SP^2 \times 4SA$, that is, (*because $4SA$ is a given Quantity.*) in a duplicate Proportion of the Distance SP reciprocally. W. W. S.

Corol. 1. FROM the three last Propositions it follows, that if any Body P be projected from

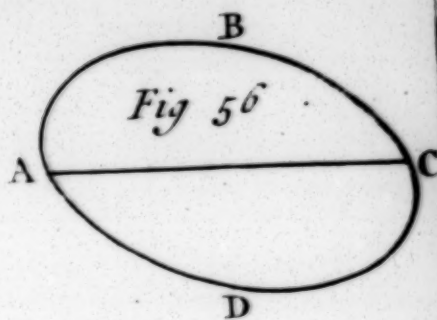
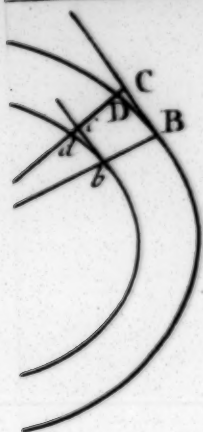


Fig. 56

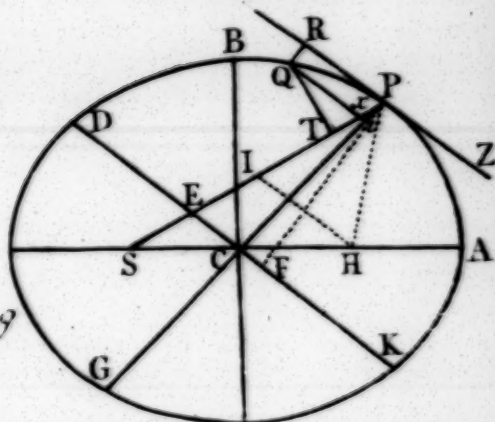
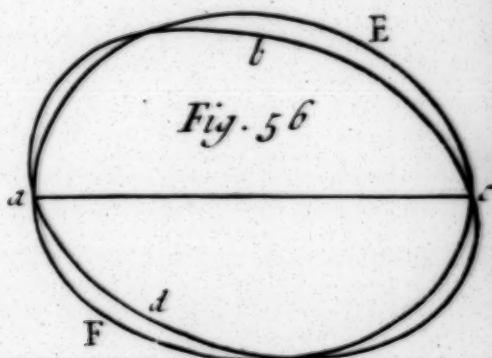


Fig. 58

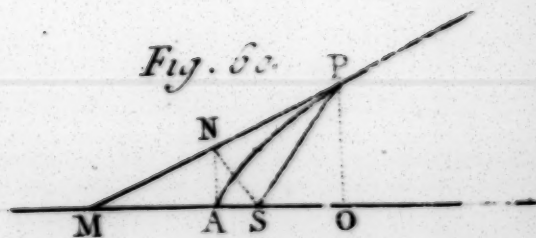
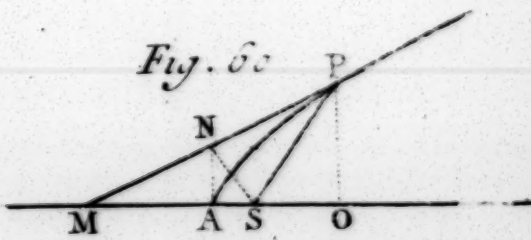
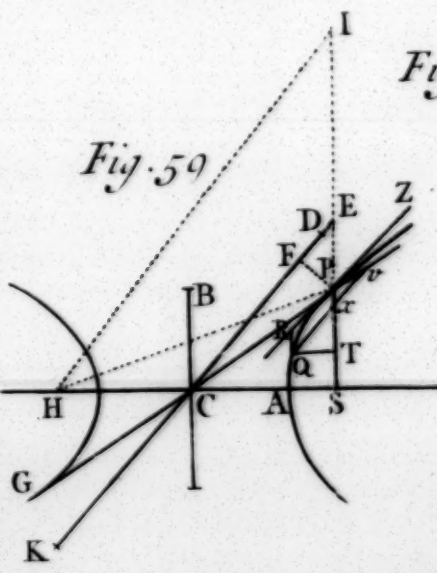
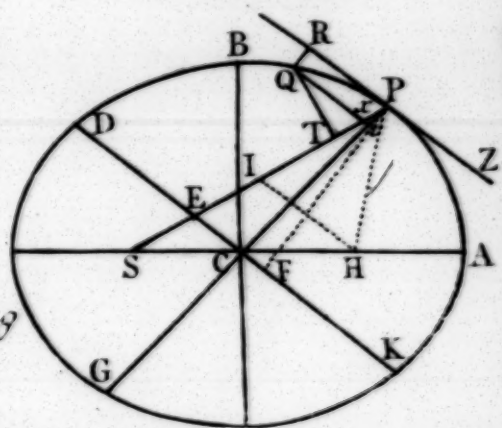
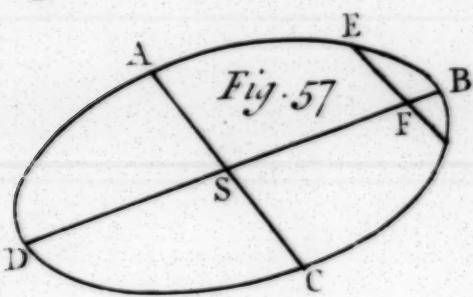
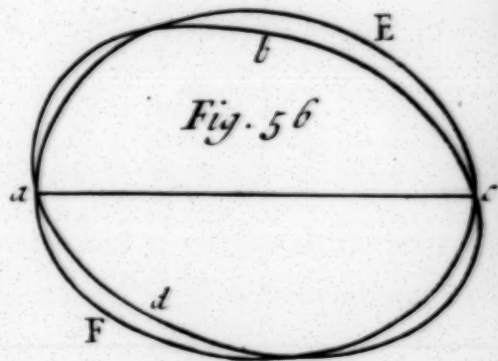
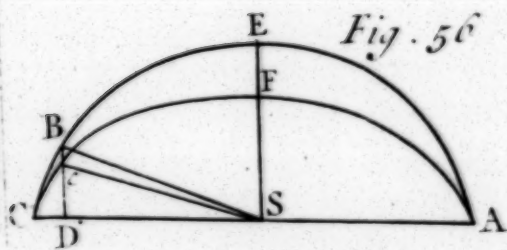
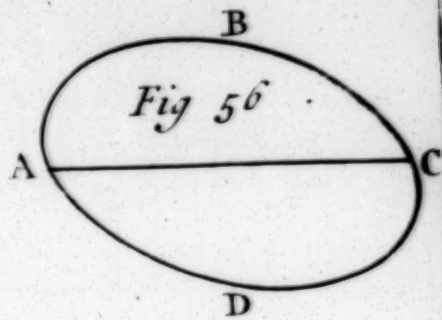
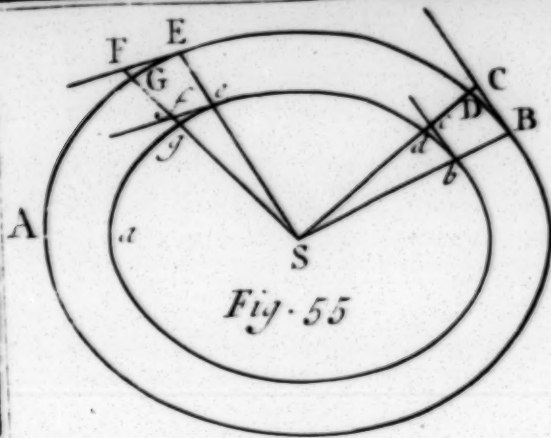
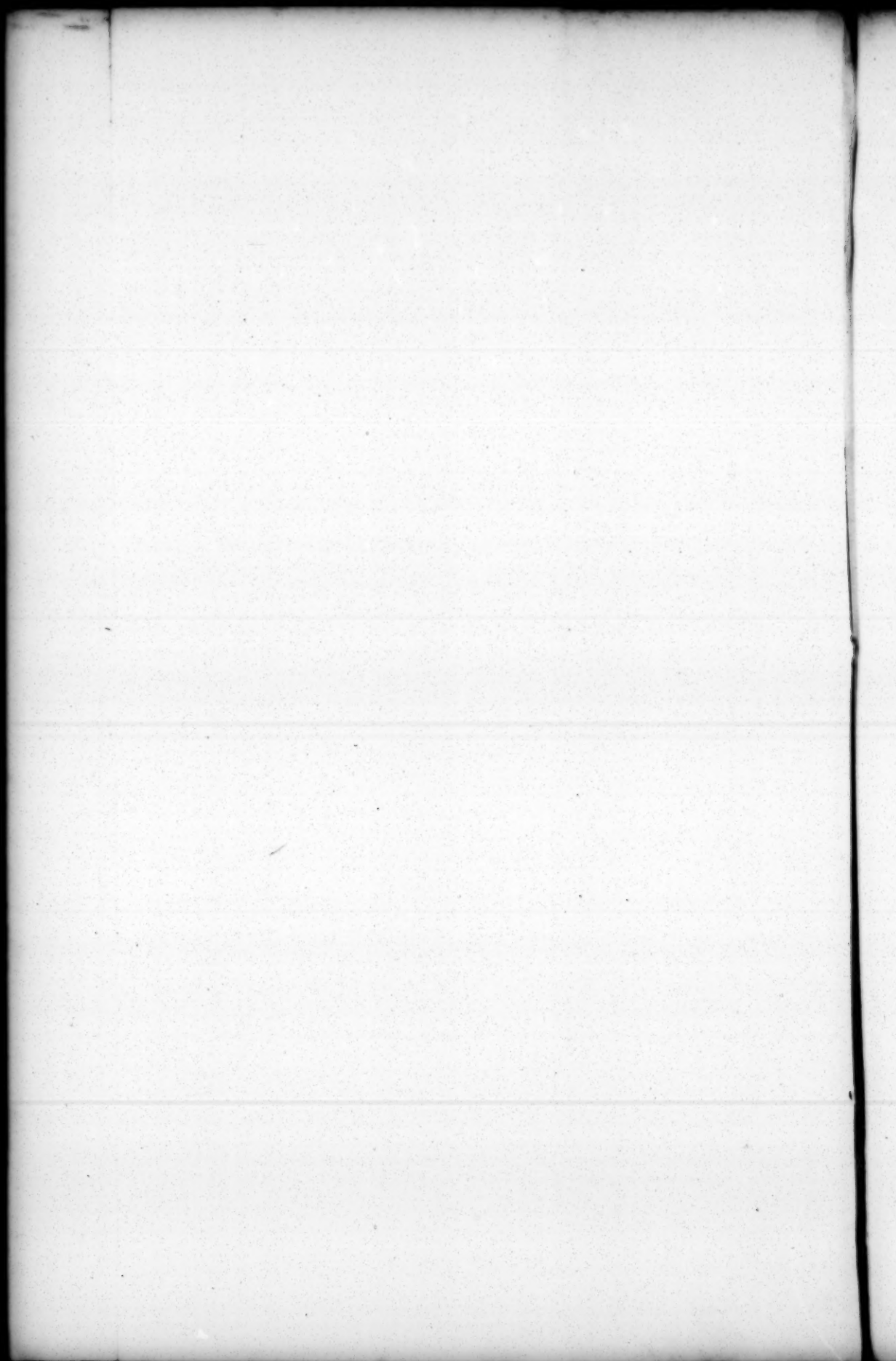


Fig. 60

it
m
y





any Point P, according to the Direction of any Line P R whatsoever, and with any Velocity; and if at the same time it be acted upon by a centripetal Force which is reciprocally proportionable to the Squares of the Distances of the Body from the Center; this Body will be moved in some one of the Conick Sections, whose Focus is in the Center of the Forces; and the contrary. For if the Focus, the point of Contact, and the Position of the Tangent be given, a Conick Section may be described which shall have a given Curvature at that Point. And the Curvature is given, if the centripetal Force and the Velocity of the Body be given. (*See these demonstrated by Dr. Keil in the Philosophical Transactions, N^o 317.*) and two Orbs touching each other cannot possibly be described with the same centripetal Force, and with the same Velocity. For let S be the Center of Forces, P the Point of Contact, O P R the Direction of the projectile Force, or the Position of the Tangent, R P the Velocity of the Body, and R Q its centripetal Force; if these be all given, P Q the remaining side of the elementary Triangle R P Q, or the infinitely small part of the Curve, is also given (*by the Elements;*) so that a Body going out from the same place, according to the same Direction, with the same Velocity, and urged by the same centripetal Force, cannot describe two different Paths. Fig. 62.

Corol. 2. IF the Velocity with which the Body is projected in the point P be such, that the Line P R may be described by it in a very small Particle of Time; and if the centripetal Force be such as to move the same Body through the Space Q R in the same Particle of Time; this Body will be moved in some one of the Conick Sections, whose principal *Latus rectum* is equal to the Quantity $\frac{QT^2}{QR}$, which happens when the small Lines P R, Q R, are infinitely diminished and vanish. In these Corollaries the Circle is to be reckoned as an Ellipsis, whose *Latus rectum* is the Diameter, and the Case where the Body descends in a streight Line to the Center is to be excepted.

S C H O L I U M.

IN the foregoing Section, where the Law of the centripetal Forces was directly as the Distances of the revolving Body from the Center, we could not apply them to any Cases in Nature, because we know of no Bodies or System which observe this Law; but if we suppose the lesser Axis of an Ellipsis in which the Body moves, to be infinitely diminished, so as its Periphery coincides with the longer Axis, the Revolution of the Body will then be turned into a Vibration backwards and forwards; and as all the Revolutions in the one Case are performed in the same Time,
so

so will all the Vibrations be in the other Case; consequently those Propositions are of excellent Use in the Calculations of pendulous Motions, and in solving several Problems about Sounds, to which Use they are afterwards put by our Author. But by the Propositions in this Section, we come to the Knowledge of the Principles of the Motions of the heavenly Bodies, and consequently what we before collected concerning these Motions from the Phenomena of Nature, may now be demonstrated *a priori*. For by applying these Propositions to the System of the Planets, it appears that they must move in Ellipses, whose Focus is in the Center of the Sun; and that if Rays be drawn from them to that Center, they will describe Areas proportionable to the Times. Thus because the Weights of the Planets upon the Sun, that is, their centripetal Forces, are reciprocally as the Squares of the Distances from the Sun; if the Sun were at rest, and the several Planets did not act upon each other, their Orbs would be elliptical, having the Sun for their common Focus, and the Areas described would be proportionable to the Times (*by Prop. I. and XI. and Cor. 1. Prop. XIII.*) But the Actions of the Planets upon each other are very small, and may therefore be neglected; for the Motion of the Planets in Ellipses about the Sun, as it moves about the common Center of Gravity, is less disturbed than if they performed their Motion about the Sun, sup-

posed to be entirely at rest. Hence we may also see the peculiar Excellency of the present Law of Gravitation, and its Suitableness in preserving the Regularity of this System the best that can be conceived. There was originally no Necessity that Gravitation should decrease as the Square of the Distance, it might have observed any other Proportion; thus for Instance, it might have decreased as the Cube of the Distance, but what would the Consequence then have been? The Planets might have been made indeed to have moved in Circles, the most regular Curves of all; but if they had acted mutually upon each other as they do now, (and as they must in all Cases do, unless we suppose Gravity not to be an universal Affection of Matter, but only occasional, and that some parcels of Matter are endued with such a Property, and others not:) The Planets, I say, could not have continued to move in those Circles, but by their mutual Action upon each other, they would have immediately disturbed one another's Orbs; and from Circles, they would have degenerated into Spirals, and either have run off from one another infinitely, or else have run into one common Center, and been mixed and confounded together: whereas by the present Law of Gravitation, they are capable of continuing for many thousand Years, without any other Disturbance or Disorder, than only a very little Alteration of the Figure of their Orbs, by the

the mutual Action of Gravity, and which can produce no ill Effect, but in a prodigious Length of Time beyond any Bounds of our Computation,

PROPOSITION XIV. THEOREM VI.

If several Bodies revolve about a common Center, and the centripetal Force be reciprocally as the Squares of the Distances of the Bodies from the Center: I say, that the principal Parameters or Latus rectum's of the Orbs, are in a duplicate Proportion of the Areas, which the Bodies describe in the same Time by Rays drawn to the Center.

FOR (by Corol. 2. Prop. XIII.) the *Latus rectum* L in the Ellipsis QP is equal to the Quantity $\frac{QT^2}{QR}$; and the *Latus rectum* l in the

Fig. 63.

Ellipsis qr , is equal to the Quantity $\frac{qt^2}{qr}$; when the points PQ , and pq , coincide, and the Lines QT , QR , and qt , qr , vanish. But the infinitely small Lines QR and qr , generated in a given Time, are as the centripetal Forces by which they are generated; that is, as SP^2 , Sp^2 reciprocally, (by the Hypothesis.) that is $\frac{1}{QR}$

is as SP^2 , and $\frac{1}{qr}$ as Sp^2 : Therefore multiplying

Concerning CENTRIPETAL FORCES.

plying each of these Quantities by QT^2 , qt^2 , it will be $\frac{QT^2}{QR} : QT^2 \times SP^2 :: \frac{qt^2}{qr} : qt^2 \times Sp^2$,

that is, putting L, l , for $\frac{QT^2}{QR} : \frac{qt^2}{qr}$; $L : l :: QT^2 \times SP^2 : qt^2 \times Sp^2$; that is, the *Latus rectum's* are to each other as the Squares of the Areas $QT \times SP$ and $qt \times Sp$. W. W. D.

Corol. HENCE it follows, that the whole Areas of the Ellipses $ABCD$, $abcd$, or the Rectangles under their Axis's, viz. $CA \times BD$, and $ca \times bd$, which are proportionable to those Areas (by *Theorem VII. Part V. of Ward's Mathematician's Guide.*) are to each other in a Proportion compounded of a subduplicate Proportion of the *Latus rectum's*, and a simple Proportion of the periodical Times; for the whole Areas are compounded of all the small Areas $QT \times SP$, and $qt \times Sp$, generated in given parts of Time; consequently the whole Areas must be as these small Areas multiplied into the periodical Times; that is, (putting A, A for the whole Areas, and P, Π for the periodical Times.) it will be $A : A :: QT \times SP \times P : qt \times Sp \times \Pi$, but $QT \times SP$, is as $\sqrt{L}$, and $qt \times Sp$ as $\sqrt{l}$, therefore (by *Substitution.*) $A : A :: \sqrt{L} \times P : \sqrt{l} \times \Pi$.

PROPOSITION XV. THEOREM VII.

The same things being supposed as before, I say that the periodical Times in Ellipses are in a sesquialterate Proportion of their longer

longer Axis's, or the Squares of the periodical Times are as the Cubes of their longer Axis's.

FOR (by Prop. XXIV. Part I. of Milnes's Fig. 63. Conick Sections.) the lesser Axis's, are a mean Proportional between the greater Axis's and the *Latus rectum's*, that is, $CA : BD :: BD : L$, and $ca : bd :: bd : l$, and (multiplying the Extremes and Means, and evolving.) $BD = \sqrt{CA \times L}$, and $bd = \sqrt{ca \times l}$; and multiplying each of these by CA , ca , it will be $CA \times BD = \sqrt{L \times CA^3}$, and $ca \times bd = \sqrt{l \times ca^3}$; that is, the Rectangles under the Axis's are in a Proportion compounded of the subduplicate Proportion of the *Latus rectum's*, and a sesquialterate Proportion of the longer Axis's. But the Rectangles under the Axis's are to each other (by Cor. Prop. XIV.) in a Proportion compounded of the subduplicate Proportion of the *Latus rectum's*, and a simple Proportion of the periodical Times; that is, $CA \times BD$ is as $\sqrt{L \times P}$, and $ca \times bd$ is as $\sqrt{l \times \Pi}$; therefore (by Substitution.) $\sqrt{L \times CA^3} = \sqrt{L \times P}$, and $\sqrt{l \times ca^3} = \sqrt{l \times \Pi}$. And taking away from each of these Equations the subduplicate Proportions of the *Latus rectum's*, that is, striking out $\sqrt{L}$, $\sqrt{l}$, there will remain $\sqrt{CA^3} = P$; and $\sqrt{ca^3} = \Pi$, that is, $P : \Pi :: \sqrt{CA^3} : \sqrt{ca^3}$. or the periodical Times are in a sesquialterate Proportion of the longer Axis's. W. W. D.

Corol.

Corol. WHEREFORE (by *Cor. 6. Prop. IV.*) the periodical Times in Ellipsis's are the same as in Circles, whose Diameters are equal to the longer Axis's of the Ellipsis's.

PROPOSITION XVI. THEOREM VIII.

The same things being supposed as before, and streight Lines drawn from the places where the Bodies are, which may be Tangents to the Orbs in those places, and also Perpendiculars let fall from the common Focus to those Tangents; I say, that the Velocities of the Bodies are in a Proportion compounded of the Proportion of those Perpendiculars inversely, and the subduplicate Proportion of the principal Latus rectum's directly.

Fig. 63. LET the Perpendiculars SY, Sy, be let fall from the Focus S to the Tangents PR, pr; and the Velocities of the Bodies P, p will be in a subduplicate Proportion of the Quantities $\frac{SY^2}{L}$ and $\frac{Sy^2}{l}$ reciprocally; or putting V for the Velocity of P, and U for the Velocity of p; it will be $V : U :: \frac{1}{\frac{SY^2}{L}} : \frac{1}{\frac{Sy^2}{l}}$. For these

Velocities are as the very small Arches PQ, pQ, described in a given Particle of Time, that is, (by *Lemma VII.*) as the Tangents PR, pr;

pr ; now because Q and R ; q and r are infinitely near each other, and the Angle at P common to the Triangles QTP , $SY P$; therefore these Triangles are similar; and for the same Reasons the Triangles qtp , Syp are similar likewise; consequently, their homologous or correspondent Sides are proportionable, that is, $PR:QT::SP:SY$, and $pr:qt::Sp:Sy$. And (by multiplying and dividing the Extremes and Means,) it will be

$$PR = \frac{SP \times QT}{SY} \quad \text{and} \quad pr = \frac{Sp \times qt}{Sy}, \quad \text{or } PR$$

is as SY reciprocally, and $SP \times QT$ directly; and pr as Sy reciprocally, and $Sp \times qt$ directly; but $SP \times QT$ and $Sp \times qt$ are as the small Areas described in given Particles of Time, that is, (by Prop. XIV.) in a subduplicate Proportion of the *Latus rectum*'s, or as $\sqrt{L}$ to $\sqrt{l}$. Wherefore if in the Proportion

$$V (=PR) : U (=pr) :: \frac{SP \times QT}{SY} : \frac{Sp \times qt}{Sy}.$$

instead of $SP \times QT$ and $Sp \times qt$, we put $\sqrt{L}$ and $\sqrt{l}$, which are proportionable to them, it will be

$$V (=PR) : U (=pr) :: \frac{\sqrt{L}}{SY} : \frac{\sqrt{l}}{Sy}. \text{W.W.D.}$$

Corol. 1. THE principal *Latus rectum*'s are in a Proportion compounded of the duplicate Proportion of the Perpendiculars, and the duplicate Proportion of the Velocities, that is, (by Transposition and Involution.) $V^2 \times SY^2 : U^2 \times Sy^2 :: L : l$

Corol. 2.

Corol. 2. THE Velocities in their greatest and least Distances from their common Focus S, that is, in c and a , C and A, are in a Proportion compounded of the Proportion of those Distances inversely, and the subduplicate Proportion of the principal *Latus rectum's* directly: For in this Case the Perpendiculars SY, Sy, become equal to the Distances SC, Sc, and SA, Sa; consequently the Proportion

$$V : U :: \frac{\sqrt{L}}{SY} : \frac{\sqrt{l}}{Sy} \text{ will become } V : u :: \frac{\sqrt{L}}{SC} : \frac{\sqrt{l}}{Sc} :: \frac{\sqrt{L}}{SA} : \frac{\sqrt{l}}{Sa} \text{ (by Substitution.)}$$

Corol. 3. AND therefore the Velocity in a Conick Section, in the greatest or least Distances SA or SC, is to the Velocity with which the Body would move in a Circle at the same Distances SA or SC from the Center, in a subduplicate Proportion of the principal *Latus rectum*, to double that Distance, that is, as $\sqrt{L}$ to $\sqrt{2SA}$. For in a Circle the Diameter, or $2SA$, is the same as the *Latus rectum* L in the Ellipsis, and may therefore be put for it in the foregoing Proportion; and then it will be (if V be put for the Velocity in the point A in the Conick Section, and U for the Velocity in the Circle described on the Center S, with the Radius SA.) $V : U :: \frac{\sqrt{L}}{SA} : \frac{\sqrt{2SA}}{SA}$, and dividing by SA, it is $V : U :: \sqrt{L} : \sqrt{2SA}$.

Corol. 4.

Corol. 4. THE The Velocities of Bodies moving in Ellipsis's, are at the mean Distances SB, Sb , from the common Focus S , the same as the Velocities of Bodies moving in Circles, whose Radius's are equal to those Distances; that is (*by Corol. Prop. IV.*) reciprocally, in a subduplicate Proportion of the Distances, or

$V : U :: \frac{1}{\sqrt{SB}} : \frac{1}{\sqrt{Sb}}$. For then the Perpendiculars SY, Sy , become equal to the lesser Semiaxis's BZ, bz ; and these Semiaxis's are mean Proportionals between the Distances, and the *Latus rectum's* (*by Prop. XXIV. Part I. of Milnes's Conick Sections.*) that is, $SB : BZ :: BZ : L$, and $Sb : bz :: bz : l$. consequently $BZ = \sqrt{L \times SB}$, and $bz = \sqrt{l \times Sb}$. Let these Proportions be multiplied inversely by the subduplicate Proportions of the *Latus rectum's* directly; that is, instead of SY, Sy , in the foregoing Proportion $V : U :: \frac{\sqrt{L}}{SY} : \frac{\sqrt{l}}{Sy}$, let $\sqrt{L} \times \sqrt{SB}$, and $\sqrt{l} \times \sqrt{Sb}$ be put, and it will become $V : U :: \frac{\sqrt{L}}{\sqrt{L} \times \sqrt{SB}} : \frac{\sqrt{l}}{\sqrt{l} \times \sqrt{Sb}}$, or striking out $\sqrt{L}$ and $\sqrt{l}$ in each, it will be $V : U :: \frac{1}{\sqrt{SB}} : \frac{1}{\sqrt{Sb}}$.

Corol. 5. IN the same Figure, or in different Figures, if the principal *Latus rectum's* be equal; the Velocities of the Body or Bodies are reciprocally as the Perpendiculars SY, Sy ,

let fall from the Focus S to the Tangents PR ,
pr. For if the *Latus rectum*'s be equal, they
 may be left out in the Proportion $V : U ::$

$$\frac{\sqrt{L}}{SY} : \frac{\sqrt{l}}{Sy}, \text{ and then it will become } V : U ::$$

$$\frac{1}{SY} : \frac{1}{Sy}.$$

L E M M A.

If an indefinite streight Line QPT touch any
 given Ellipsis or Hyperbola any where in P ,
 and from either of the Focus's S or F , sup-
 pose S , the Perpendicular SY be let fall to
 the Tangent, and SP, FP be drawn; I say,
 that the Perpendicular SY is in a subduple-
 cate Proportion of SP to FP , or SY is as

$$\sqrt{\frac{SP}{FP}}.$$

Fig. 64. FROM the other Focus F , let another Per-
 pendicular FT be let fall to the same Tangent,
 and let it meet SP produced in C ; then the
 streight Line TO , which bisects FS in the
 Center O , will also bisect FC in T , because
 the Angles TPF and TPC are equal, and FP
 is equal to PC , (*by Prop. III. Part IV. of*
Milnes's Conick Sections;) and the same Line
 TO will also be parallel to SC , and equal to
 half of it, (*because the Triangles FTO and*
FCS are similar.) or equal to half the Sum or
 Difference of the streight Lines SP, PF , or
 equal

equal to the principal Semiaxis OA or OB (by Prop. V. Part IV. of Milnes's Conick Sections.) If TO therefore be produced till it meet SY (produced also) in t , a Circle described on the Center O, with the Radius OB, will pass through the Points B, T, t , A, Y, (because BO, TO, AO, t O, were before shown to be all equal to each other, and T t is a Diameter of the Circle, and the Triangle TY t a right-angled Triangle inscribed in the Semicle) wherefore the Rectangle t S $\times$ SY, or FT $\times$ SY, is every where equal to the given Rectangle AS $\times$ SB. Now because

$$FT \times SY : SY^2 :: FT : SY. \text{ (by equal Multiplication.)}$$

and FT : SY :: FP : SP. for the right-angled Triangles SYP, FPT are similar, because the Angles SPY, FPT, are equal (by Prop. III. Part IV. of Milnes's Conick Sections.)

Wherefore FT $\times$ SY : SY² :: FP : SP. by Division.

O

And

And $\frac{SY^2}{FT \times SY} = \frac{SP}{FP}$ by Multiplication and Transposition.

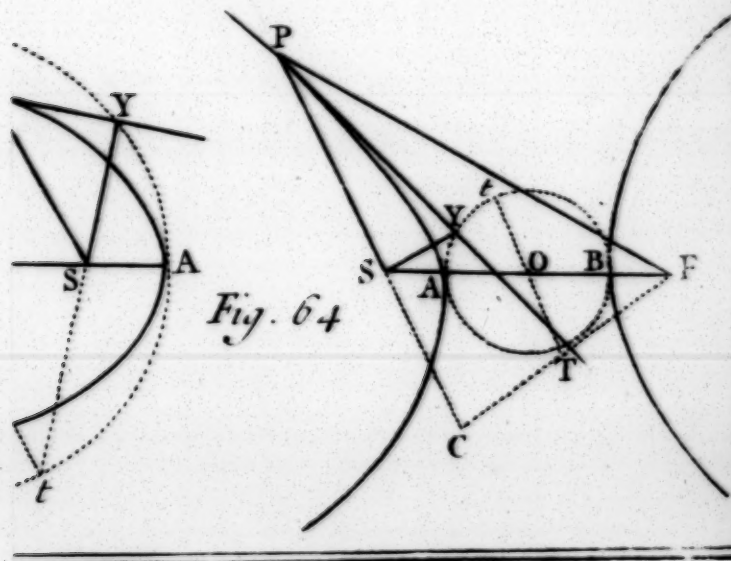
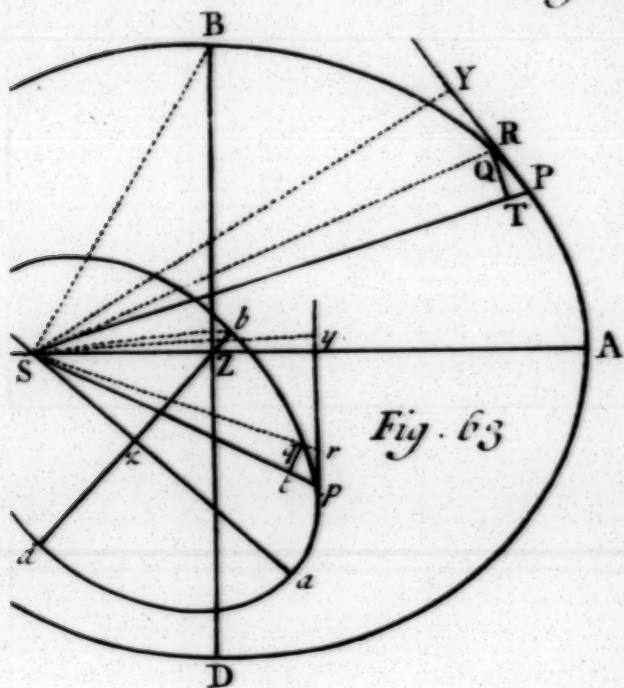
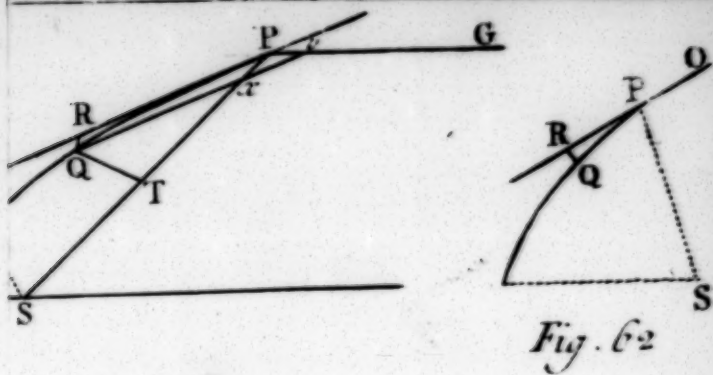
And because $FT \times SY$ is a given Quantity, SY^2 is as $\frac{SP}{FP}$, and SY as $\sqrt{\frac{SP}{FP}}$ (by Evolution.)

W. W. D.

Corol. 6. IN a Parabola, the Velocity of the Body is every where reciprocally in a subduplicate Proportion of its Distance from the Focus of the Figure: In the Ellipsis the Proportion is greater than this, in the Hyperbola it is less. For (by *Cor. 2. Lemma XIV.*) a Perpendicular let fall from the Focus of the Parabola to the Tangent, is in a subduplicate Proportion of the Distance, that is, $SY : Sy :: \sqrt{SP} : \sqrt{Sp}$. Wherefore if in the Proportion of the last Corollary, viz. $V : U :: \frac{1}{SY} : \frac{1}{Sy}$, instead of SY, Sy ; we put $\sqrt{SP}, \sqrt{Sp}$; which are proportionable to them; it will be $V : U :: \frac{1}{\sqrt{SP}} : \frac{1}{\sqrt{Sp}}$. But in the Ellipsis and Hyperbola the Perpendiculars SY, Sy , are as $\sqrt{\frac{SP}{FP}}, \sqrt{\frac{Sp}{Fp}}$, (by the last Lemma.) Therefore in these Figures it is $V : U :: \frac{1}{\sqrt{\frac{SP}{FP}}} : \frac{1}{\sqrt{\frac{Sp}{Fp}}}$.

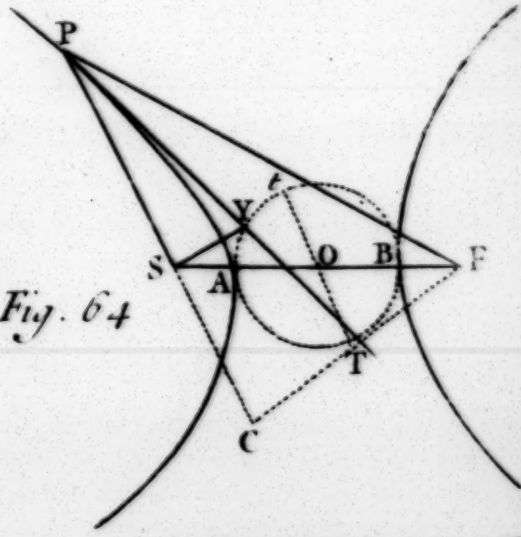
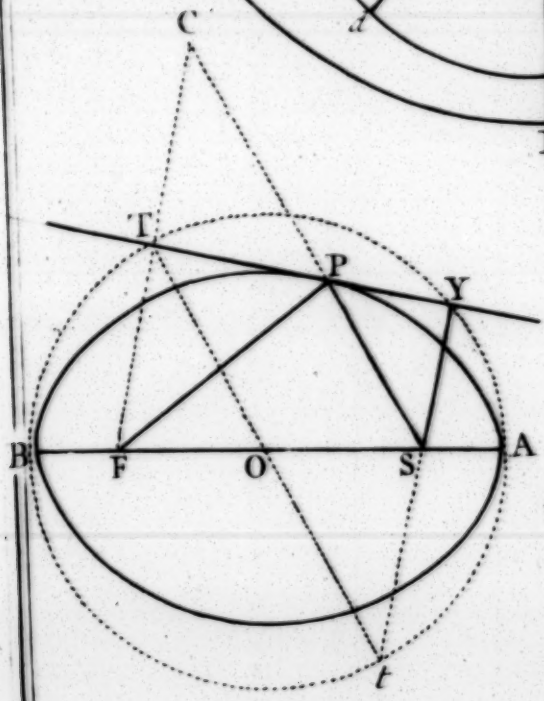
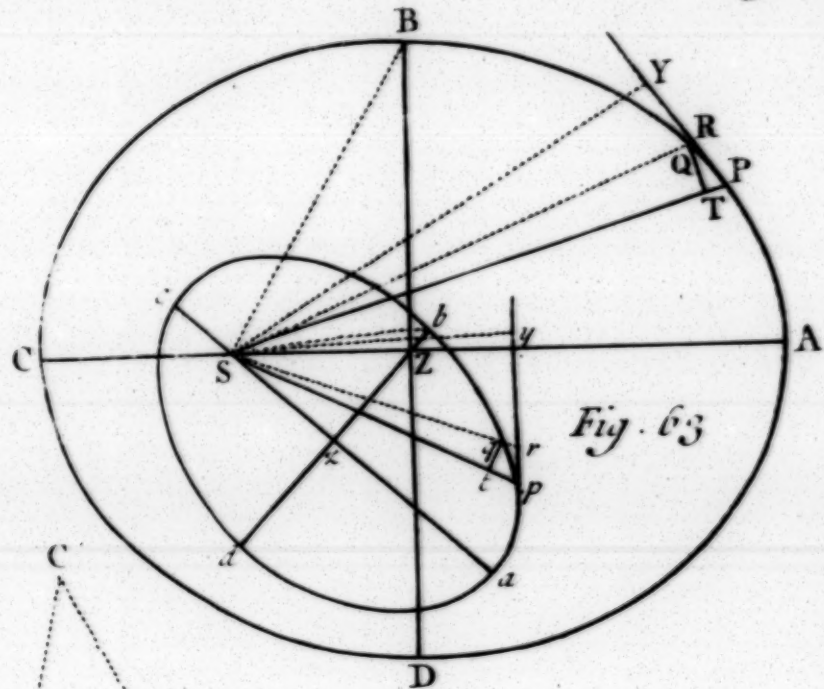
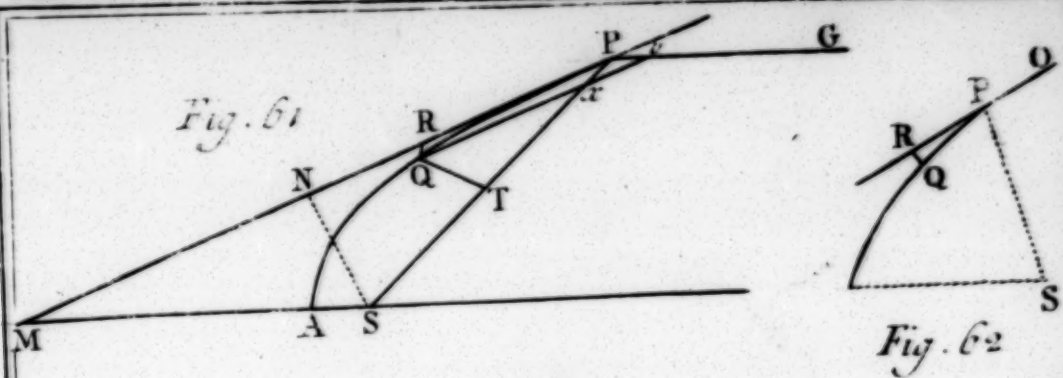
So that the Perpendicular SY undergoes less Alteration in the Hyperbola than in the Parabola, and more in the Ellipsis.

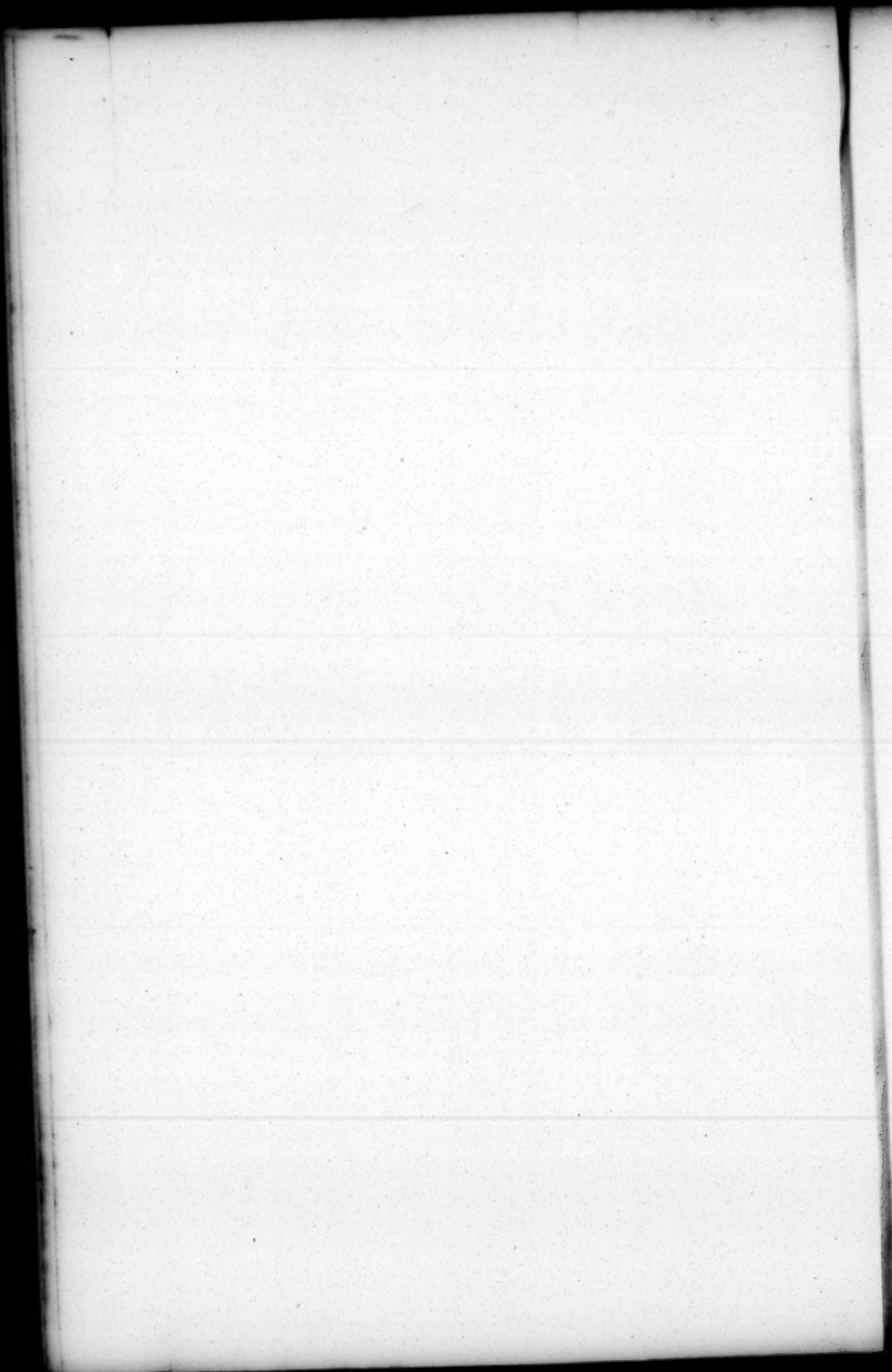
Corol. 7.



ne
li-
us
on
efs.
u-
to
on
P:
of
in-
ich
U::

Hy-
as
fore
I
Sp
Fp
less
ara-





Corol. 7. IN the Parabola, the Velocity of Fig. 66. the Body P, at any Distance PS from the Focus S, is to the Velocity of a Body revolving in a Circle, whose Radius is the same Distance PS, in a subduplicate Proportion of the number Two to Unity; in the Ellipsis the Proportion is less than this, and in the Hyperbola it is greater. For (*by the 2d and 3d Cor. of this Prop.*) the Velocity in the Vertex A of the Parabola, is to the Velocity in a Circle, whose Radius is SA; as $\frac{\sqrt{L}}{SA}$ to $\sqrt{2SA}$; but (*by Prop. II. Part IV. of Milnes's Conick Sections.*) L is equal to 4SA, therefore those Velocities are to each other as $\sqrt{\frac{4SA}{SA}}$ to $\sqrt{2SA}$, that is, as $\sqrt{2}$ to 1 (*by Division.*) and the same Proportion holds for any other Distance SP; for (*by the 6th Corol. of this Prop. and the 6th Cor. of the IVth Prop.*) the Velocity at the Distance SP in the Parabola, is to the Velocity in a Circle whose Radius is SP, as $\frac{1}{\sqrt{SP}}$ to $\frac{1}{\sqrt{SP}}$; that is, they are every where in the same Proportion to each other; consequently, if they are to one another as $\sqrt{2}$ to 1 in the Vertex, they are so at all other Distances. That the Proportion is less than this in the Ellipsis, and greater in the Hyperbola, will appear in Corol. 9. Hence it also follows, that the

Velocity in the Parabola, is every where equal to the Velocity of a Body revolving in a Circle, whose Radius is half the Distance; for the Velocity at P in the Parabola, is to the Velocity in a Circle whose Radius is SP, as $\sqrt{2}$ to 1, and the Velocity in this Circle is to the Velocity in a Circle whose Radius is $\frac{1}{2}$ SP, as 1 to $\sqrt{2}$ (by Cor. 6. Prop. IV.) Therefore they are equal (by Equality of Proportion;) for let V be put for the Velocity of the Body at P in the Parabola, C for the Velocity in a Circle whose Radius is SP, and v for the Velocity in a Circle whose Radius is $\frac{1}{2}$ SP. Then it will be

$$V : C :: \sqrt{2} : 1.$$

$$\text{and } C : c :: 1 : \sqrt{2}.$$

$$\text{Therefore } V : c :: \sqrt{2} : \sqrt{2}.$$

that is $V = C$.

In the Ellipsis this Velocity is less, in the Parabola it is greater, as you will find by Cor. 9.

Corol. 8. THE Velocity of a Body revolving in any Conick Section, is to the Velocity of a Body revolving in a Circle whose Radius is equal to half the principal *Latus rectum* of the Section, or equal to $\frac{1}{2}$ L, is as that Radius to the Perpendicular let fall from the Focus to the Tangent of the Section, or as $\frac{1}{2}$ L to SY. For (by Corol. 5.) though the Figures be different, yet if the *Latus rectums* be equal, the Velocities are reciprocally as the Perpendiculars let fall from the Focus to the Tangents,

that is, $V : U :: \frac{1}{SY} : \frac{1}{Sy}$. But if $\frac{1}{2}L$ be equal to the Radius, the Diameter of the Circle, or its *Latus rectum* is equal to the *Latus rectum* of the Ellipsis, and the Perpendicular Sy becomes equal to the Radius $\frac{1}{2}L$; therefore $V : U :: \frac{1}{SY} : \frac{1}{\frac{1}{2}L}$ (by Substitution.) or $V : U :: \frac{1}{2}L : SY$.

Corol. 9. HENCE it follows, that because the Velocity with which a Body revolves in this Circle, is to the Velocity with which a Body revolves in any other Circle, in a subduplicate Proportion of their Distances or Radius's reciprocally; (by *Cor. 6. Prop. IV.*) that is, supposing C to be the Velocity in this last Circle, and D its Radius, $U : C :: \frac{1}{\sqrt{\frac{1}{2}L}} :$

$\frac{1}{\sqrt{D}}$. Hence I say it follows (by Composition.) that the Velocity of a Body revolving in a Conick Section, is to the Velocity of a Body revolving in a Circle at the same Distance, as a mean Proportional betwixt that common Distance and half the principal *Latus rectum* of the Section, to the Perpendicular let fall from the common Focus of the Section to its Tangent. For if we multiply the corresponding Terms of the Proportion in the last Corollary, viz. $V : U :: \frac{1}{2}L : SY$. into the Terms

of the Proportion in this, viz. $U : C :: \frac{1}{\sqrt{\frac{1}{2}L}}$:

$\frac{1}{\sqrt{D}}$. and divide by the common Term U ; it

will be $V : C :: \frac{\frac{1}{2}L}{\sqrt{\frac{1}{2}L}} : \frac{SY}{\sqrt{D}}$. But $\frac{\frac{1}{2}L}{\sqrt{\frac{1}{2}L}} = \sqrt{\frac{1}{2}L}$; (by Division.) consequently (by Substitution and Transposition.) it will be, $V : C :: \sqrt{\frac{1}{2}L \times D} : SY$. And these Proportions computed in Numbers, will appear to be less than $\sqrt{2}$ to 1 in the Ellipsis, and more than the $\sqrt{2}$ to 1 in the Hyperbola.

PROPOSITION XVII. PROBLEM IX.

Supposing that the centripetal Force is reciprocally proportionable to the Square of the Distances from the Center, and that the absolute Quantity of this Force is known: It is required to find the Line described by a Body going out from a given Place with a given Velocity, and according to the Direction of a given streight Line.

Fig. 67. LET the known centripetal Force tending to the point S be such, as that the Body p may be revolved by it in any given Orb. And let the Velocity of this Body be known in the place p . Let the Body P go out with a given Velocity from the place P , according to the Direction of the Line PR , and let it be immediately diverted from its rectilinear Course by

by the centripetal Force, and turned into the Conick Section PQ. Then will the Line PR touch this Section in P. Let some other streight Line pr touch the Orb pq in p , and if from S there be let fall Perpendiculars to these Tangents, then (*by Cor. 1. Prop. XVI.*) the principal *Latus rectum* of the Conick Section, will be to the principal *Latus rectum* of the Orb, in a Proportion compounded of the duplicate Proportion of the Perpendiculars, and the duplicate Proportion of the Velocities, and consequently the principal *Latus rectum* of the Conick is given; let it be L. The Focus S of this Conick Section is given also. And because (*by Prop. III. Part IV. of Milnes's Conick Sections.*) if the Line PH pass through the other Focus H, the Angles RPS and TPH will be equal, therefore if the Angle RPH be the Complement of the Angle RPS to two right Angles, the Line PH, in which the other Focus H is, will be given in Position (*by the Elements.*) Let SK be let fall perpendicular to PH, and suppose the conjugate Semiaxis BC to be erected, then (*by Prop. XIII. Book II. of the Elements.*) $SP^2 - 2 KP \times PH + PH^2 = SH^2 = 4 CH^2$ (because $SH = 2 CH$) $= 4 BH^2 - 4 BC^2$ (*by the Elements.*) $= SP \times PH^2 - L \times SP \times PH$ (because $SP + PH = 2 BH$, and L is a third Proportional to $SP + PH$ and $2 BC$, *by Prop. XXIV. Part I. and Prop. V. Part IV. of Milnes's Conick Sections.*) $= SP^2 + 2 SP \times PH + PH^2 - L \times SP + PH$ (*by multiplying*

tipling $SP + PH$ into $SP + PH$.) let $2KP \times PH - SP^2 - PH^2 + L \times \overline{SP + PH}$ be added on each side of the Equation $SP^2 - 2KP \times PH + PH^2 = SP^2 + 2SP \times PH + PH^2 - L \times \overline{SP + PH}$; and it will be, $L \times \overline{SP + PH} = 2SP \times PH + 2KP \times PH$, or by resolving the Equations into Proportion $SP + PH : PH :: 2SP + 2KP : L$. consequently PH , which was before given in Position, is now given in Length also. So that if the Velocity of the Body in P be such, that the *Latus rectum* L is less than $2SP + 2KP$, then PH will be a finite Quantity, and consequently will lie on the same side of the Tangent PR , as the Line PS does; and then the Figure must be an Ellipsis, and the two Focus's S, H , and the principal Axis being given, it may easily be described (by Prop. VI. Part V. of Milnes's *Conick Sections*.) But if the Velocity of the Body in P be so great, that the *Latus rectum* L be equal to $2SP + 2KP$; then PH will become equal to $SP + PH$, that is, SP will be infinitely small with respect to PH , or the Distance between P and H will be infinite; and because the *Latus rectum* L is equal to $2SP + 2KP$; therefore AS , the Distance of the principal Vertex from the Focus, is equal to $\frac{2SP + 2KP}{4}$,

Fig. 68.

(by Prop. II. Part IV. of Milnes's *Conick Sections*.) and the Figure may be described by Prop. VII. Part V. of Milnes's *Conick Sections*. And if the Body goes out from the place P with

with a still greater Velocity, so that the *Latus rectum* L be greater than $2 SP + 2 KP$, then Fig. 69. PH will be greater than $SP + PH$, that is, it will be more than infinite, or it will lie on the other side of the Tangent PR , and therefore the Tangent being situate between the two Focus's, the Figure will be a Hyperbola, having its principal Axis equal to the Difference of the two Lines SP, PH , and will therefore be given. For the principal *Latus rectum*, or L , and the Focus S of the Conick Section being given, if the Angle RPH be made equal to the Angle RPS , the Line PH , in which the other Focus H is placed, is given (*by Prop. III. Part IV. of Milnes's Conick Sections.*) Let SK be let fall perpendicular from S to PH , and suppose the conjugate Semiaxis BC to be erected, then (because $KH = HP - KP$, and $SH^2 = KH^2 + KS^2$, and $KS^2 = SP^2 - KP^2$) $SP^2 - 2 KP \times PH + PH^2 = SH^2 = 4 CH^2 = 4 BH^2 - 4 BC^2 = PH - PS^2 - L \times PH - PS$ (*by Part IV. Prop. V. of Milnes's Conick Sections.*) $= PS^2 - 2 SP \times PH + PH^2 - L \times PH - PS$ (by multiplying $PH - PS$ into $PH - PS$.) let there be added to each side of this Equation $2 KP \times PH - SP^2 - PH^2 + L \times PH - PS$, and the Equation will become $L \times PH - PS = 2 SP \times PH - 2 KP \times PH$, or (*resolving this Equation into Proportion.*) as $PH - PS$ is to PH , so will $2 SP - 2 KP$ be to L , and the Figure may be described *by Prop. VI. Part V. of Milnes's Conick Sections.* For if in these
several

several Cases, a Body be revolved in a Conick Section, found in this manner, it is demonstrated in the XIth, XIIth, and XIIIth Propositions, that the centripetal Force will be reciprocally as the Square of the Distance of the Body from S, the Center of the Forces; and therefore the Line PQ is rightly represented, and would be described by that Body, if it goes out of the given place P, with a given Velocity, according to the Direction of the straight Line PR given in Position. W. W. D.

Corol. 1. HENCE it follows, that if the principal Vertex D, the *Latus rectum* L, and the Focus S, be given in any Conick Section, the other Focus H will be given also, if we take DH to DS, as the *Latus rectum*, to the Difference between the *Latus rectum* and 4 DS. For in the Case of this Corollary, where P is supposed to coincide with D, the Proportion of $SP + PH : PH :: 2 SP + 2 KP : L$. will become $DS + DH : DH :: 4 DS : L$. consequently by Division $DS : DH :: 4 DS - L : L$.

Corol. 2. HENCE the Orb may very speedily be found, if the Velocity in the principal Vertex D be given. For the Velocity with which the Body would revolve in a Circle, whose Radius is the Distance DS, is given (*by Corol. 11. and Scholium of Prop. IV.*) for it is equal to the Velocity generated by the given centripetal Force in the Time that the Body would fall through half the Distance DS. And (*by Cor. 3. Prop. XVI.*) the *Latus rectum* L is had by taking

taking it in the same Proportion to twice the Distance DS ; as the Square of the given Velocity in the Conick Section, is to the Square of the Velocity of the Body revolving in a Circle, whose Radius is PS . Then take DH to DS , as the *Latus rectum* to the Difference betwixt the *Latus rectum* and $4DS$, and the Figure may be described by *Prop. VI. Part V. of Milnes's Conick Sections.*

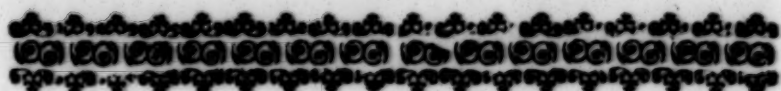
Corol. 3. HENCE it follows also, that if a Body be moved in any Conick Section whatsoever, such as $p q$ suppose, and be forced out of its Orb by any Impulse whatsoever; the Orb PQ , in which it will afterwards move, may be found; for if the Velocity generated by the Impulse singly, be added to the Velocity which the Body had before, we shall have the Velocity with which the Body goes out from the given place where the Impulse was made, according to the Direction of a streight Line given in Position, which is the Case of this Proposition.

Corol. 4. AND if the Body be continually put out of its Course by some external Force impressed upon it; its future Course may be very nearly found out, by computing the Alteration made by that Force in several places, and so judging of the continual Alteration in the intermediate places by the Proportion which the Terms of the Series bear to each other.

S C H O L I U M.

IF a Body P be moved in the Perimeter of Fig. 70. any given Conick Section whatsoever, whose Center is C, by a centripetal Force tending to any given point R; and it be required to find the Law of the centripetal Force; let CG be drawn parallel to the Ray RP, and meet the Tangent of the Orb PG in G; and the Force will be proportionable to $\frac{CG^3}{RP^2}$. For if the centripetal Force C tend to the Center C, it will be proportionable to the Distance CP (*by Cor. 1. and the Scholium to Prop. X.*) and this Force will be to the Force c tending to the point R, as $CP \times RP^2$, to CG^3 (*by Cor. 3. Prop. VII.*) that is, $CP : c :: CP \times RP^2 : CG^3$. and multiplying the Extremes and Means, and dividing by CP, it will be c , or the Force tending towards R, proportionable to $\frac{CG^3}{RP^2}$.

SECTION



SECTION VII.

*Concerning the Ascent and Descent of Bodies in
streight Lines directly.*

PROPOSITION XXXII. PROBLEM XXIV.

Let the centripetal Force be supposed to be reciprocally proportionable to the Square of the Distances of the places where any Body is, from the Center; it is required to determine the Spaces described by such a Body falling directly down in a streight Line, in any given Times.

Case I. **I**T was before demonstrated (*Prop. XIII. Corol. 1.*) that under these Circumstances, the Body must either fall down perpendicularly, or it must describe one of the Conick Sections, either some Circle, or Ellipsis, or Parabola, or Hyperbola, whose Focus is coincident with the Center of the Forces. Let *ARPB* be that Conick Section, and let *S* Fig. 71. be the Focus of it; and first let us suppose this Figure to be an Ellipsis, then let the Semicircle *ADB* be described upon the Line *AB*, which is the longer Axis of the Ellipsis; and as the Body falls down, let the streight Line *DPC* pass through it when it is in the place

place C, and be perpendicular to the Axis AB. Now if the streight Lines DS and PS be drawn, the Area ASP will be proportionable to the Time in which the Body would move in the Ellipsis from A to P (*by Prop. I.*) but the Area ASD is proportionable to the Area ASP (*by Prop. V. Book V. of L'Hospitals Conick Sections.*) and consequently the Area ASD is proportionable to the Time in which the Body would move in the Ellipsis from A to P. Let the Axis AB remain the same, and let the Breadth of the Ellipsis be continually diminished; the Proportion which is betwixt these two Areas will be always the same, and the Area ASD will always be proportionable to the Time, though the Breadth of the Ellipsis become never so small. Let the Breadth of it be diminished infinitely, so that the Distance CP may be less than any given Distance whatsoever; then will the Orb APB coincide with the Axis AB, and the Focus S will coincide with the Extremity of the Axis B, and the Body instead of moving in an Ellipsis, will descend in the streight Line AC, which in this Case may be conceived to be an Ellipsis of no given Breadth, or whose shorter Axis is infinitely small; and the Area ABD will still be proportionable to the Time in which the Body descends from A to C in this streight Line, or infinitely diminished Ellipsis. If therefore the Area ABD be taken proportionably to the Time, and the Perpendicular DC be let fall from the

the

the point D to the streight Line AB; the Space AC will be given, which the Body describes in a given Time, by falling perpendicularly from the place A. W. W. S.

Case 2. If the Figure RPB be an Hyperbola, then let the rectangular Hyperbola BED (that is, an Hyperbola, whose longer and shorter Axis's or principal Diameters intersect each other at right Angles) be described upon the same principal Diameter AB (by *Prop. VI. Part V. of Milnes's Conick Sections*;) then because the Areas CSP, CBfP, SPfB; and the Areas CSD, CBED, SDEB, are all in the same Proportion to each other, that is, in the given Proportion of the Lines CP, CD, or CSP is to CSD, as CP to CD, and so of the rest. (See *L'Hospital's Conick Sect. Book V. Prop. VI.*) and because the Area SPfB is proportionable to the Time in which the Body P moves through the Arch PFB (by *Prop. I.*) therefore the Area SDEB will be proportionable to the same Time also, and may be put to represent it. Now let the *Latus Rectum* of the Hyperbola RPB be infinitely diminished, and the *Latus Transversum*, or the Line AB, remain the same; and the Hyperbola RPB will become infinitely narrow, so that the Curve PB will coincide with the streight Line CB, the Focus S will also coincide with the Vertex B, and the streight Line SD with the streight Line BD; consequently the BDEB will be proportionable to

Fig. 72.

the Time in which the Body C by descending directly downwards, describes the streight Line CB: So that while the Body C is descending, all the Lines CB may be looked upon as so many infinitely narrow Areas of an infinitely narrow Hyperbola corresponding, and proportionable to as many Areas BDEB, described by the Point D, in forming the Hyperbola BED, and also corresponding and proportionable to the several equal Moments of Time in which they are both generated. W. W. S.

Fig. 73. *Case 3.* The same Reasoning holds likewise, if the Figure RPB be a Parabola: for let another Parabola BED be described on the same principal Vertex B, (*by Prop. VII. Part V. of Milnes's Conick Sections.*) and the Area CSP will be to the Area CSD as CP to CD, (*by the Elements;*) the Area CBfP will be to the Area CBED, in the same Proportion of CP to CD, because the Areas of Parabolas are equal to two third Parts of their circumscribed Parallelograms, (*by Prop. XIII. Book V. of L'Hospital's Conick Sections.*) and these Parallelograms being of the same height, are as their Bases CP, CD (*by the Elements.*) and therefore (*by Division.*) the Area SPfB, is to the Area SDEB, as CP to CD. Let the Parabola BED continue the same, till the other Parabola RPB, in whose Perimeter the Body P moves, coincides with the Line CB, which it will do, when its *Latus rectum*, by being infinitely diminished, is reduced to nothing;

thing; and then the Parabolick Segment BDEB, which is proportionable to the Time in which the Body P or C descends to the Center S or B, will also be proportionable to the Line CB; and therefore if the Area BDEB be cut off proportionable to the Time, and from the Point D, the Line DC be let fall perpendicular to the Axis CB, and intersect it in C; the Space AC will be described by the falling Body in the Time represented by the Area BDEB. W. W. S.

SCHOLIUM.

By the help of this Proposition, we may Fig. 74. easily find how long Time any Planet will take in falling directly to the Center of the Sun by the Force of Gravitation only, if its projectile Motion in its Orbit were stopped. Let S be the Sun, ARB the Orbit of the Planet A; it is manifest (*by this last Proposition.*) that in the same Time that the Planet A revolves in its Orbit from the Point A to the Point B, it would by the force of Gravity fall directly through the Diameter from A to B; for the same Reason, if the same Planet A revolved about the Center C, in the Orbit AKS, it would fall from A to S, by the Force of its own Gravity, in the same time that it would revolve in the Orbit AKS from A to S, that is, if the same Planet be supposed to describe different Orbits, the Times of its direct Descent through the Diameters of
P those

those Orbits, are to each other as half the periodical Revolutions; but (*by Prop. XV.*) half the periodical Revolutions are to each other in a sesquialterate Proportion of the Distances, that is, in this Case, half the periodick Revolution in the Orbit A R B, is to half the periodick Revolution in the Orbit A K S, in a sesquialterate Proportion of A B to A S, that is, (because A S is equal to half A B) as $\sqrt{8}$ to 1; and because half these periodical Revolutions are equal to the direct Descents from A to B, and from A to S, therefore half the periodick Revolution of the Planet A, in the Orbit A R B, is to the Time of its direct Descent from A to S, as $\sqrt{8}$ to 1. and the whole Time of the periodick Revolution, is to the Time of the direct Descent to the Center C, as $2\sqrt{8}$ to 1, or as $4\sqrt{2}$ to 1. Now $4\sqrt{2}$ is equal to 5,656; therefore if the Time of the periodical Revolution of any Planet about the Sun be divided by 5,656, the Quotient will give the Time that the same Planet would take up in falling directly down to the Center of the Sun. Thus it will be found that the Earth and the Moon together, would arrive at the Sun in the Space of about 66 Days and 19 Hours, *Venus* would arrive there in about 40 Days, *Mercury* in about 15 Days and a half, *Jupiter* in the Space of two Years and one Month, and so for the rest. If the Moon were stopped in its Orb, it would descend to the Earth in about five Days, by the same Method of Computation.

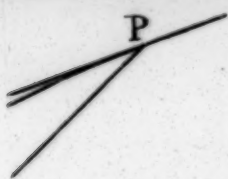


Fig. 65

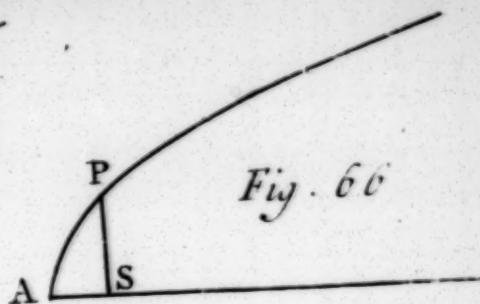


Fig. 66

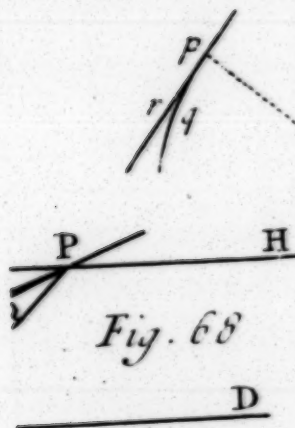


Fig. 68

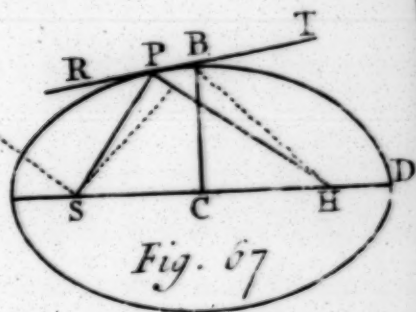


Fig. 67

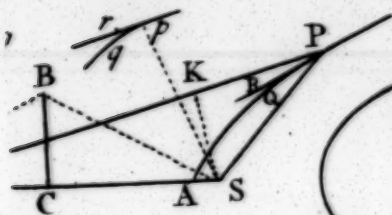


Fig. 73

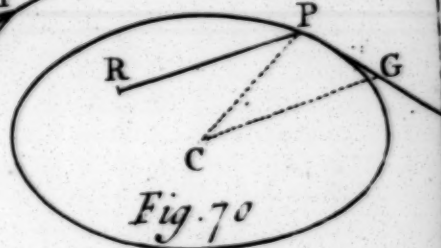


Fig. 70

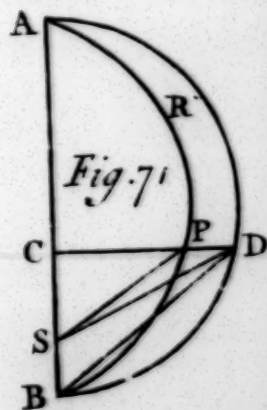
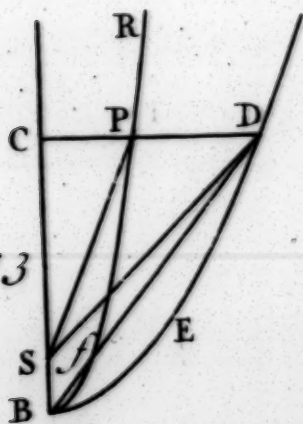


Fig. 71

e-
lf
in
es,
o-
e-
a
is,
I;
ons
B,
ck
B,
to
he
he
I,
to
cal
di-
me
ing
ns
oon
ace
uld
in
ace
the
it
ays,
o-

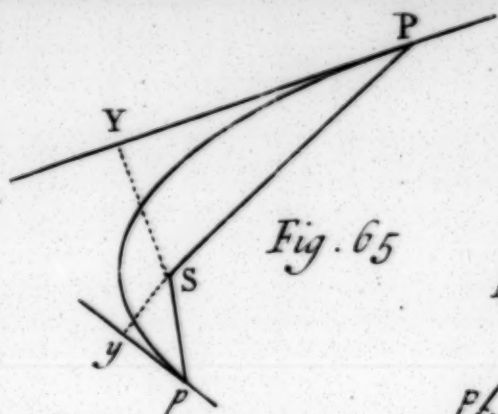


Fig. 65

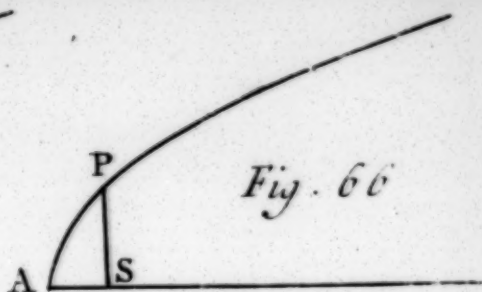


Fig. 66

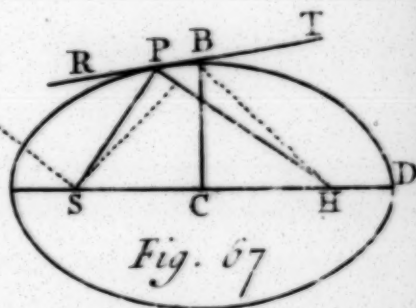


Fig. 67

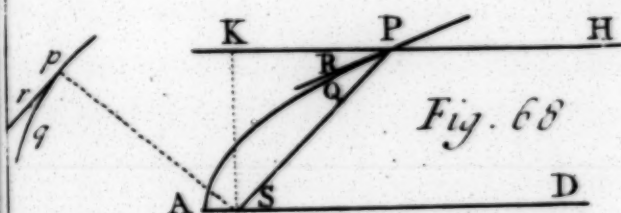


Fig. 68

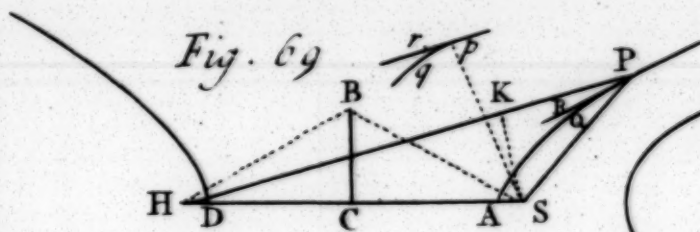


Fig. 69

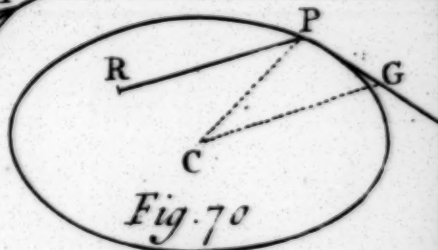


Fig. 70

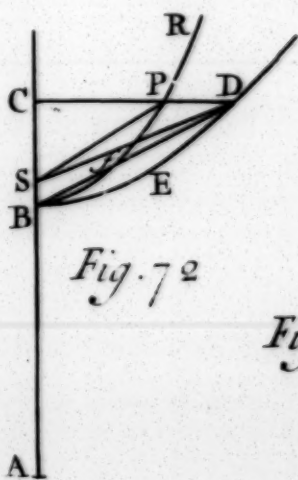


Fig. 72

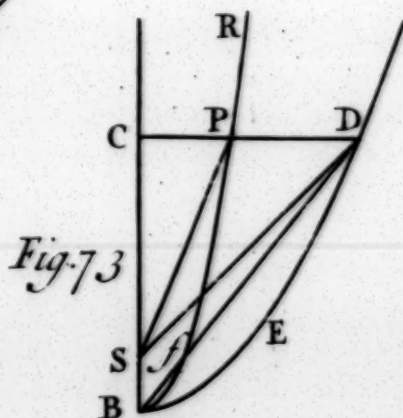


Fig. 73

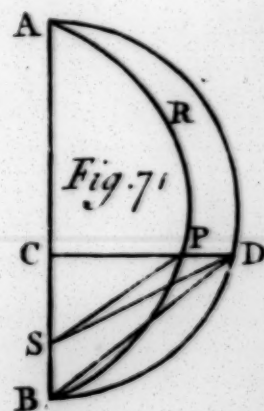


Fig. 71

PROPOSITION XXXIII. THEOREM IX.

The things demonstrated in the foregoing Propositions being supposed, I say that the Velocity of a falling Body in any place C, is to the Velocity of a Body describing a Circle about the Center B, at the Distance BC, in a subduplicate Proportion of AC to $\frac{1}{2}$ AB; that is, as the square Root of the Distance of the Body from A, the furthest Vertex of the Circle or rectangular Hyperbola, to the square Root of the principal Semidiameter of the Figure.

LET AB, the common Diameter of both the Figures RPB, DEB, be bisected in O, Fig. 75. and let the streight Line PT be drawn, which may touch the Figure RPB in P, and may also intersect the common Diameter AB (produced, if need be) in T; let SY be drawn perpendicular to this Tangent, and BQ, perpendicular to the Diameter AB, and let the *Latus rectum* of the Figure be supposed equal to L. It is evident by the 9th Corol. of the XVIth Prop. that the Velocity of a Body moving in the Line RPB, about the Center S, is in any place P, to the Velocity of a Body describing a Circle about the same Center at the Distance SP, in a subduplicate Proportion of the Rectangle $\frac{1}{2}$ L $\times$ SP to SY square: that is, putting U for the Velocity

P 2
city

Concerning CENTRIPETAL FORCES.

city in the Conick Section, and V for the Velocity in the Circle;

$$U: V :: \sqrt{\frac{1}{2}L \times SP}: SY.$$

but $ACB:CP^2:: 2AO : L$ (by Part I. Prop. 24. Cor. 5. of Milnes's Conick Sections.)

Therefore $\frac{2 CP^2 \times AO}{ACB} = L$ (by Transposition.)

Therefore putting L for its Value in the foregoing Proportion, it will be,

$$U: V :: \sqrt{\frac{CP^2 \times AO \times SP}{ACB}}: SY.$$

Further,

$$CO: BO :: BO: TO. \text{ (Milnes's Conick Sections, Part II. Prop. I. Cor. 2.)}$$

And

$$CO: BO :: CB: BT. \text{ (by Composition or Division.)}$$

wherefore

$$BO \mp CO: BO :: CT: BT. \text{ (by Division or Composition,)} \\ \text{that is,}$$

$$AC: AO :: CP: BQ; \text{ (because the Triangles TCP, TBQ, are similar.) consequently}$$

$$CP^2 = \frac{BQ^2 \times AC^2}{AO^2} \text{ (by Multiplication and Transposition.)}$$

And if in the Quantity $CP^2 \times$

$\frac{CP^2 \times AO \times SP}{ACB}$, instead of CP^2 be substituted

its Equivalent $\frac{BQ^2 \times AC^2}{AO^2}$; the Result will be

$$\frac{CP^2 \times AO \times SP}{ACB} = \frac{BQ^2 \times AC \times SP}{AO \times BC}.$$

Now let CP , the breadth of the Figure RPB , be infinitely diminished, so that the Point P may coincide with the Point C ; the Point S with the Point B ; the Line SP with the Line BC , and the Line SY with the Line BQ ; and the Proportion will be in that case,

$$U : V :: \sqrt{\frac{BQ^2 \times AC \times SP}{AO \times BC}} : SY. \text{ or}$$

$$U : V :: \sqrt{BQ^2 \times AC \times SP} : \sqrt{SY^2 \times AO \times BC}.$$

that is, the Velocity of the Body descending directly downwards in the streight Line CB , will be to the Velocity of a Body describing a Circle about the Center B , and at the Distance CB ; in a subduplicate Proportion of $BQ^2 \times AC \times SP$ to $SY^2 \times AO \times BC$, that is, (if the Terms which are equal to each other in this Case, *viz.* SP, BC , and BQ^2, SY^2 , be struck out of the Proportion) these Velocities will be to each other in a subduplicate Proportion of AC to AO , or $\frac{1}{2} AB$. W. W. D.

Corol. 1. When the Points B and S coincide, TC will be to TS as AC to AO . for it was proved before, that $AC : AO :: CP : BQ$. but $CP : BQ :: TC : TS$ (because the Triangles TCP , TSQ , or TBQ , are similar;) therefore $TC : TS :: AC : AO$.

P 3

Corol. 2.

Corol. 2. If a Body revolves in any Circle at a given Distance from the Center, and its Motion were turned upwards, it would ascend to twice its Distance from that Center, and then return back again. For suppose B the Center of the Circle, and BO the given Distance from this Center, and the Point C, where the falling Body is, to coincide with the Point O: Then, by this Proposition, the Velocity of the falling Body, will be to the Velocity of the revolving Body, in a subduplicate Proportion of AO to AO, that is, these Velocities will be equal; but the falling Body descended from A to O, consequently the revolving Body, whose Velocity is the same, would ascend to A, if the Determination of its Motion were upwards, that is, to twice the Distance of OB from the Center B.

SCHOLIUM.

HENCE we may observe the different Effects of the Force of Gravitation, upon Supposition that this Force continues the same at all Distances from the Center; and upon Supposition that it decreases in a duplicate Proportion of the Distance. In the former Case, the Velocity with which a Body revolves in any Circle at a given Distance from the Center, is exactly equal to the Velocity acquired by the Force of Gravity in falling directly through a Space equal to half the Radius of the Circle
(by

(by Prop. IV. Cor. 9.) in the latter Case, the Velocity of the revolving Body is exactly equal to the Velocity acquired by the Force of Gravity in falling directly through a Space equal to the whole Radius of the Circle; that is, just double the Length of the other.

PROPOSITION XXIV. THEOREM X.

If the Figure BED be a Parabola, I say, that the Velocity of the falling Body in any Place C, is equal to the Velocity with which a Body may be made to describe a Circle on the Center B, with half the Distance BC, with an uniform Motion.

FOR the Velocity of a Body describing the Fig. 76. Parabola RPB in any place P, is equal to the Velocity of a Body uniformly describing a Circle about the same Center S, at half the Distance SP. (by Cor. 7. Prop. XVI.) Let CP, the breadth of the Parabola, be infinitely diminished, so that Pfb, the Arch of the Parabola, may coincide with the streight Line CB, the Center S, with the Vertex B, and the Distance SP, with the Distance BC; then will the Velocity in C be equal to the Velocity of a Body describing a Circle about the Center B, with an uniform Motion, at the Distance of half CB. W. W. D.

PROPOSITION XXXV. THEOREM XI.

The same things being supposed, I say, that the Area of the Figure DES, described by the indefinite Radius SD, is equal to an Area which may be described in the same Time by a Body moving uniformly about the Center S, in a Circle whose Radius is equal to half the Latus rectum of the Figure.

Fig. 77. For suppose the Body C to describe as it falls, the very small Line Cc in the least Particle of Time that can be; and another Body K in the same Moment of Time to describe the Arch Kk, as it revolves uniformly in the Circle OKk, about the Center S. Let the Perpendiculars CD, cd, be erected, and meet the Figure DES in D and d. Let SD, Sd; SK, Sk, be joined; and let Dd be produced till it meets the Axis AS in T, and let SY be drawn perpendicular to it.

Case I. Now if the Figure DES be a Circle, or a rectangular Hyperbola, let the transverse Diameter AS be bisected in O, and SO will be half the Diameter of the Circle, or half the *Latus rectum* of the Hyperbola (by Part I. Prop. XXIV. Cor. 1. of Milnes's *Conick Sections*.) And

because

because $TC : TD :: Cc \text{ or } dg : Dd$, because the

Triangles
 TCD, dgD
are similar (by
the Elements.)

And $TD : TS :: CD : SY$, because the

Triangles
 TCD and
 SYT are
similar, for
they are
right-angled
Triangles,
and the An-
gle at T is
common.

There- $TC : TS :: CD \times Cc : SY \times Dd$. (by equal
fore Multiplica-
tion.)

But $TC : TS :: AC : AO$ (by Corol. I.
Prop.

XXXIII.)
if the last
Proportion
of these
Lines be ta-
ken when the
Points D and
 d coincide.

There- $AC : AO \text{ or } SK :: CD \times Cc : SY \times Dd$. (by
fore Equality of
Proportion.)

Further,

Further, the Velocity (U) of the descending Body in C, is to the Velocity (V) of a Body describing a Circle about the Center S, at the Distance SC, in a subduplicate Proportion of AC to AO, or SK, (*by Prop. XXXIII.*)

that is, $U : V :: \sqrt{AC} : \sqrt{SK}$. and the Velocity (U) in this Circle, is to the Velocity (Λ) of a Body describing the Circle OKk, in a subduplicate Proportion of SK to SC (*by Corol. 6. Prop. IV.*)

that is, $V : \Lambda :: \sqrt{SK} : \sqrt{SC}$. Therefore the first Velocity (U) is to the last Velocity (Λ .)

that is, $U : \Lambda :: \sqrt{AC} : \sqrt{SC}$, or the very small Line Cc, which is proportionable to the Velocity of a falling Body in C, is to the Arch Kk, which is proportionable to the Velocity of the revolving Body in K, these small Spaces being supposed to be described in the same Moment of Time; in a subduplicate Proportion of AC to SC, or in the Proportion of AC to CD (*by Prop. XIII. Book VI. of the Elements*) that is, $Cc : Kk :: AC : CD$. Wherefore (*by multiplying the Extremes and Means*) $CD \times Cc$ is equal

equal to $AC \times Kk$. Wherefore if in the foregoing Proportion of $AC : SK :: CD \times Cc : SY \times Dd$, instead of $CD \times Cc$, we put its Equal $AC \times Kk$, it will be, as AC is to SK , so will $AC \times Kk$ be to $SY \times Dd$. and (*multiplying the Extremes and Means, and dividing by AC ,*) it will be, $SK \times Kk$, equal to $SY \times Dd$; and (*by Division.*) $\frac{1}{2} SK \times Kk$, equal to $\frac{1}{2} SY \times Dd$; that is, the Area $KS k$ is equal to the Area $SD d$. (*by Schol. Prop. XLI. Book I. of the Elements.*) In every Moment of Time therefore the Particles $KS k$ and $SD d$ of two Areas are generated; and if the Magnitude of them be diminished, and the Number of them increased infinitely, they will become equal to each other, and consequently the whole Areas composed of an equal infinite Number of equal infinitely small Areas, will always be equal likewise (*by Corol. 3. Lemma IV.*) W. W. D.

Fig. 78.

Case II. If the Figure DES be a Parabola, by the very same way of demonstrating, it will appear that in this Case also $CD \times Cc$ will be to $SY \times Dd$, as TC is to TS ; but by the Property of the Parabola TC is to TS as 2 to 1, (*Milnes's Conick Sect. Part II. Prop. II.*) Therefore $\frac{1}{4} CD \times Cc$ is equal to $\frac{1}{2} SY \times Dd$, (*by multiplying the Extremes and Means, and by Schol. Prop. XLI. Book I. of the Elements.*) Now the Velocity (U) of the descending Body in the Point C , is equal to the Velocity (V) with which a Body may be made to move uniformly in a Circle, at the Distance of $\frac{1}{2} SC$ from the Center S (*by Prop. XXXIV.*) And this

Concerning CENTRIPETAL FORCES.

this Velocity (V ,) is to the Velocity (Λ ,) with which a Body may be made to describe a Circle with the Radius SK , that is, the very small Line Cc , which is proportionable to the Velocity U or V ; is to the Arch Kk , which is proportionable to the Velocity Λ (for these small Spaces are supposed to be described both in the same Moment of Time;) in a subduplicate Proportion of SK to $\frac{1}{2} SC$ (by *Corol. 6. Prop. IV.*) that is,

$Cc : Kk :: \sqrt{SK} : \sqrt{\frac{1}{2} SC}$; that is, $:: SK : \frac{1}{2} CD$. For, by the Hypothesis, $2 SK$ is equal to the *Latus rectum* of the Parabola, and $CD^2 = 2 SK \times SC$ (by *Part I. Prop. XXV. of Milnes's Conick Sections*,) therefore $\sqrt{\frac{CD^2}{4 SK}} = \sqrt{\frac{1}{2} SC}$, and (by *Substitution and Division*.)

$$Cc : Kk :: SK : \frac{1}{2} CD.$$

Wherefore $\frac{1}{2} SK \times Kk$ is equal $\frac{1}{4} CD \times Cc$. (by *multiplying the Extremes and Means*, and *dividing by 2.*) and it was before shown, that $\frac{1}{4} CD \times Cc$ is equal to $\frac{1}{2} SY \times Dd$. consequently (by *Substitution*) $\frac{1}{2} SK \times Kk$ is equal to $\frac{1}{2} SY \times Dd$; that is, the Area $KS k$ is equal to the Area $SD d$, as in the foregoing Proposition. *W. W. D.*

PROPOSITION XXXVI. PROBLEM XXV.

How to determine the Times of the Descent of a Body falling from a given place A.

LET the Semicircle ADS be described upon
 Fig. 79. the Diameter AS , the Distance of the Body from the Center S , when it begins to fall; and
 on

On this Center S, let another Semicircle OKH be described equal to the former one. From any place C, where the Body is as it falls, let the Ordinate applicate CD be drawn. Join SD, and take the Sector OSK equal to the Area ASD. Then it is evident (*by Prop. XXXV.*) that the Body in falling will describe the Space AC, in the same Time that another Body may be made to describe the Arch OK, by revolving uniformly about the Center S; for SO, the Radius of this latter Circle, is equal to half the *Latus rectum* of the Figure ADS, or half the Diameter AS. W. W. D.

PROPOSITION XXXVII. PROBLEM XXVI.

How to determine the Times of the Ascent and Descent of a Body thrown upwards or downwards from a given place.

LET a Body be thrown from any given place G, according to the Direction of the Line GS, and with any Velocity (U) whatsoever. Let GA be to $\frac{1}{2}$ AS, in a duplicate Proportion of this Velocity (U) to the uniform Velocity (V,) with which a Body may be made to revolve about the Center S, at the given Distance SG, that is, Fig. 80.

$GA : \frac{1}{2} AS :: U^2 : V^2.$ (*by Prop. XXXIII.*)

Case I. IF this Proportion be that of the Numbers 2 to 1, that is, if $U^2 : V^2 :: 2 : 1$, then the Body will ascend or descend by the same Law of Velocity, as if it moved in a Para-

bola

bola (*by Prop. XXXIV.*) consequently the Point A will be at an infinite Distance, because the Axis of the Parabola is infinite. In this Case therefore, there must be a Parabola described (*SDI* suppose) on the Vertex S, and with any *Latus rectum* whatsoever. Then on the Center S with the Distance SH, equal to half the *Latus rectum*, let the Circle HkK be described; and at the place G, where the ascending or descending Body is, let the perpendicular GI be erected, meeting the Parabola in I, and at any other place C, let the Perpendicular CD be erected also, meeting the Parabola in D. Again, let SI and SD be joined, and let the Sector HSK, HSk, be made equal to the Segments SEIS, SEDS, and then, (*by Prop. XXXV.*) the Body G will describe the Space GC in the same Time that the Body K may be made to describe the Arch Kk. W. W. D.

- Case 2 and 3. IF the Proportion of U^2 to V^2 be less than that of 2 to 1, in this Case the Figure to be described must be a Circle, whose Diameter is SA; and if the Proportion of U^2 to V^2 be greater than that of 2 to 1, then the Figure to be described must be a rectangular Hyperbola, whose principal Diameter is SA (*by Prop. XVI. Cor. 7. and Prop. XXXIII.*) Upon the Center S, with the Distance of half the Diameter SA in Case the 2d; or with the Distance of half the *Latus rectum* of the Hyperbola SEID, in Case the 3d, let the Circle HIK

He
use
his
de-
nd
on
to
be
the
er-
ra-
the
the
be
be
S,
will
hat
rch

V^2
the
ose
 U^2
the
ular
S A
II.)
half
the
Hy-
rcle
IK

$\frac{D}{z}$

3.

the
ause
this
de-
and
on
to
be
the
per-
ara-
the
the
be
be
S,
will
ha:
rch

V²
the
ose
U²
the
lar
A
(I.)
alf
he
y-
le
K



Fig. 74

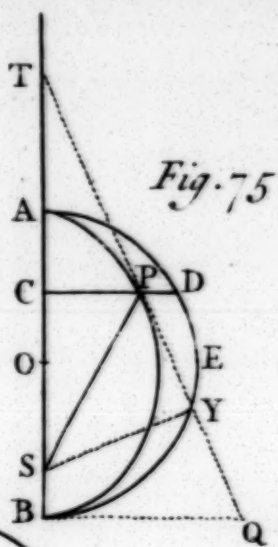


Fig. 75

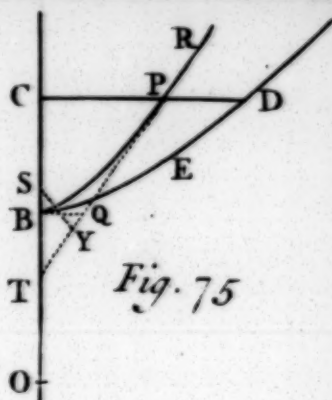


Fig. 75

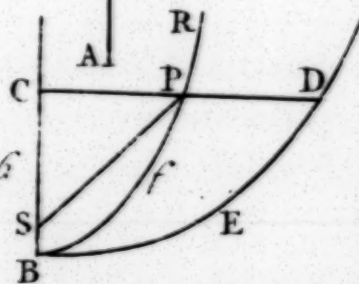


Fig. 76

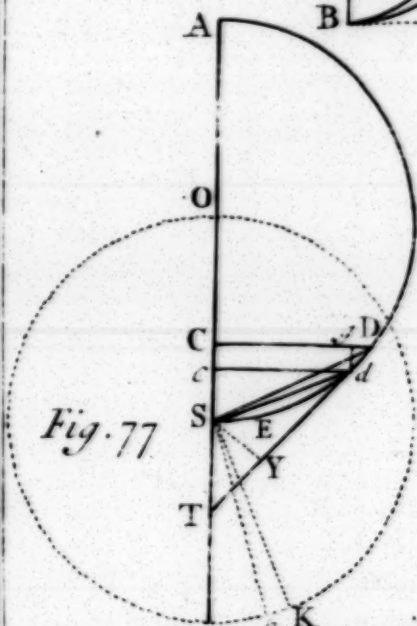


Fig. 77

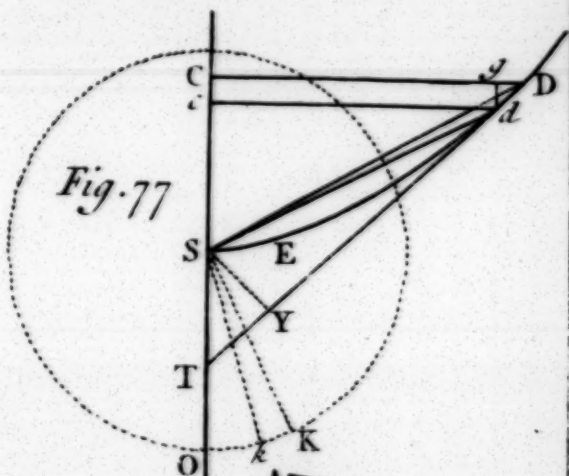


Fig. 77

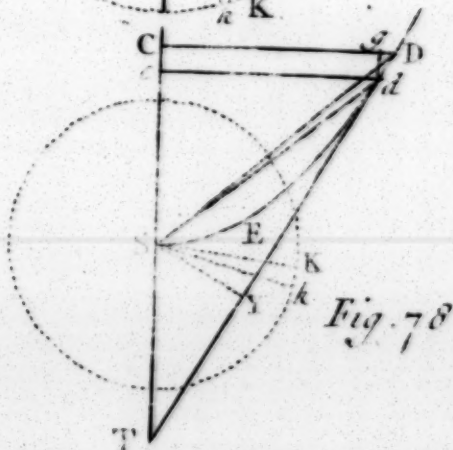


Fig. 78

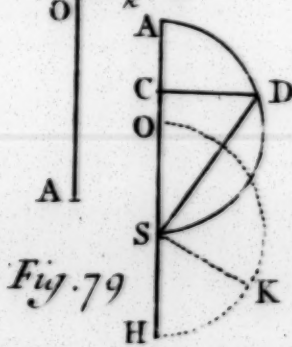


Fig. 79

H
th
P
If

c
u
r
C
n
v
n
L
v

HIK be described; and the rest will be exactly the same as in Case the 1st. W. W. D.

PROPOSITION XXXVIII. THEOREM XII.

If we suppose the centripetal Force to be proportionable to the Height or Distance of the places where the falling Body is from the Center; I say, that the Times of its Fall, are proportionable to the Arches of a Circle described on a Radius equal to the Distance of the falling Body from the Center: The Velocities of the falling Body are proportionable to the right Sines of those Arches, and the Spaces which it falls through, are proportionable to the versed Sines of the same Arches.

LET the Body fall from any place A, according to the Direction of the Line AS; then upon S, the Center of the Forces, and with the Distance AS, let the Quadrant of the Circle AE be described; and let CD be the right Sine of any Arch AD, and AC, the versed Sine of the same Arch; then will AD represent the Time of the Body's falling from A to C; AC will represent the Space it falls through; and CD will represent the Velocity which it has acquired in falling from A to C.

IF the Body A does not descend in the streight Line AS towards S, or move in the Circle ADE, it will move in some Ellipsis,
whose

Fig. 83.

whose Center is S , the Center of the Forces, (by *Cor. 1. Prop. X.*) Let ARP be this Ellipsis, and let S be the Center of it; then if the streight Lines DS and PS be drawn, the Area ASP will be proportionable to the Time in which the Body would move in the Ellipsis from A to P (by *Prop. I.*) but the Area ASD is proportionable to the Area ASP , (by *Prop. V. Book V. of L' Hospital's Conick Sections.*) and therefore the Area ASD is proportionable to the Time in which the Body would move in the Ellipsis from A to P . Let the breadth of the Ellipsis be infinitely diminished, so that the upper Focus may coincide with the Point A , and the Orb ARP with the streight Line ACS , and the Body, instead of moving in the Orb ARP , will descend directly to the Center S , in the streight Line ACS , and the Area ASD will still continue proportionable to the Time in which the Body moves from A to C , in the streight Line AC . Wherefore if the Area ASD be taken proportionable to the Time, and from the Point D , the Perpendicular DC be let fall to the Semiaxis AS , the Space AC will be described by the falling Body in the Time represented by the Area ASD ; but the Area ASD is proportionable to the Arch AD (by the *Elements*;) and therefore the Arch AD is proportionable to the Time of Bodies descending through the Space AC , which is the versed Sine of that Arch. Further, let the Line cd be drawn as near as can be to CD ,
and

and from the point d , let de be let fall perpendicular to CD ; then will cC be equal to de ; and because the very small Space cC , is described in the given Time dD , therefore the Velocity of the falling Body in C , is proportionable to this small Space cC ; (for if the Time be given, the Velocity is as the Space described,) but the Triangles deD and CDS are similar (for they are right-angled at e and C ; and the Angles $CDS + DSC$ are equal to a right Angle, the Angles $CDS + dDe$ are also equal to a right Angle, therefore the Angle DSC is equal to the Angle dDe ;) consequently dD is to de or cC , as SD to CD ; but SD , the Radius of the Circle, is given, and therefore CD is proportionable to cC ; that is, to the Velocity in C . W.W.D.

Corol. 1. HENCE it follows, that a Body in falling from the place A , will arrive at the Center S , at the same time that another Body, by revolving in the Circle AE , will describe the Quadrant AE . For the Velocity of the revolving Body being uniform, both the Velocity and the Time may be represented by the given Space dD , which being correspondent all the way to cC , when C is got to S , D must be got to D (*by Corol. Lem. IV.*)

Corol. 2. HENCE it follows also, that all Bodies, from what different Heights so ever they fall, will arrive at the Center at the same time. For if they revolved about the Center, their periodical Times would be all the same

Q

(by

(by Corol. III. Prop. IV.) Therefore by the foregoing Corol. their Times of Descent will be all the same.

SCHOLIUM.

IF there were a Hole bored in any part of the Earth's Surface quite through the Center to the opposite Surface; and any Number of Bodies let fall at the same time at different Distances, some very near the Center, and some very near the Surface; they would all arrive at the Center together, and then separate, and get as far from the Center on the other side, as they were when first let fall; after which they would all at the same time begin to come back to the Center again, and arrive at it all of them together; and thus they would uniformly vibrate backward and forward continually: For the centripetal Force within the Earth's Surface is proportionable to the Distance from the Center, as shall be shown at Prop. LXXIII.

PROPOSITION XXXIX. PROBLEM XXVII.

It is required to find the Velocity of a Body ascending or descending directly, in any place where it is; and also the Time in which the Body will arrive at any place; let the centripetal Force be of any kind whatsoever; (that is, let it increase or decrease, according to any Law whatsoever :) it being supposed only that

curve-lined Figures can be squared: And the contrary.

LET the Body E fall from any place A, in the streight Line ADEC; and in whatever place E it is, let the Perpendicular EG be always erected proportionable to the centripetal Force in that place, by which it tends to the Center C: And let BFG be the curve Line continually described by the Point G. Further, at the beginning of the Motion, let the Line EG coincide with the Perpendicular AB, which represents the centripetal Force in A, and the Velocity of the Body in any place E will be in a subduplicate Proportion of the curve-lined Area ABGE. W.W.S.

IN the Line EG, let EM be taken in a reciprocal Proportion of the square Root of the Area ABGE; and let VLM be the curve Line which the Point M touches in every place as the Line EM moves from A to E, and whose Asymtote is the streight Line AB produced: Then will the Time which the Body takes up in falling through the Space represented by the Line AE, be proportionable to the curve-lined Area ATVME. W.W.S.

DEMONSTRATION.

LET the very small Line DE of a given Length, be taken in the Line AE; that is, let Fig. 84. the Line AE be divided into an infinite Num-

Q₂

ber

ber of equal Parts DE, and let DLF be the place of the Line EMG, when the falling Body was in the Point D; then if the centripetal Force be such, that a streight Line, whose Square is equal to the Area ABGE, is proportionable to the Velocity of the descending Body; The Area it self will be in a duplicate Proportion of the Velocity (*by Involution;*) that is, if V be put for the Velocity of the Body when it is in D, and I be put for the Increment of the Velocity, while the Body falls through the infinitely small Space DE; $V + I$ will represent the Velocity of the Body when it is in E, and the Area ABFD will be proportionable to VV , and the Area ABGE will be proportionable to $\overline{V + I^2}$, that is, $VV + 2VI + II$ (*by the Hypothesis.*) But the Area DFGE, is equal to the Area ABGE — the Area ABFD, or $VV + 2VI + II - VV$; that is ($+VV - VV$ destroying each other) the Area DFGE, is proportionable to $2VI + II$; and therefore $\frac{DFGE}{DE}$ is proportionable to $\frac{2V + II}{DE}$, (*each being divided by the common Quantity DE.*) that is, if the first Proportions of the Quantities be taken as they begin to be generated, the Line DF will be proportionable to the Quantity $\frac{2VI}{DE}$; for when the Point D coincides with the

Point

Point E, the very small Space DE vanishes, and consequently (by Lemma I.) the small Area DFG E becomes equal to the Line DF. So

likewise in the Quantity $\frac{2VI + II}{DE}$, V repre-

sents the whole Velocity acquired in falling from A to D, and I represents only the small Increment of this Velocity, generated in a very short Time, whilst the Body fell through the infinitely small Space DE; II therefore the Square

of this Increment, is infinitely less than $\frac{2VI}{DE}$;

and consequently may by the same Lemma be left out, because it makes no Alteration in the Proportion; wherefore DF is proportionable

to $\frac{2VI}{DE}$, or which is the same thing, DF is proportionable to half this Quantity, or to $\frac{V \times I}{DE}$. Now the centripetal Force, with

which the Body tends towards C, is every where proportionable to the Increment of the Velocity in a given Time; because the Increase of the Velocity is the Effect of this Force; and if the Time be not given, then the centripetal Force is as the Increment of the Velocity directly, and the Time inversely, (because the more the Velocity is increased, and the shorter the Time is in which it is increased, the greater must the Force be:) Therefore let F be put for the centripetal Force, and

Q₃

T for

T for the Time; then F will be proportionable to $\frac{I}{T}$. Again, the Time in which the Body by falling describes the small Line DE, is as the Space through which it falls directly, and the Velocity inversely; that is, as $\frac{DE}{V}$: Now if in the foregoing Quantity $\frac{I}{T}$, instead of T, we substitute the Quantity $\frac{DE}{V}$, which is proportionable to it; we shall have F proportionable to $\frac{I \times V}{DE}$. Wherefore a Centripetal Force proportionable to DF, or EG, will cause a Body to descend with a Velocity which is every where in a subduplicate Proportion of the Area ABGE. W:W.D.

Further, Because all the very small Lines DE, are equal to each other, therefore the Time in which any of them is described by the falling Body, is as the Velocity inversely (for the slower any Body moves, the longer Time does it take up) that is, in the inverse Proportion of a streight Line equal to the square Root of the Area ABFD (*by the Hypothesis*;) let DL be in the inverse Proportion of that streight Line; then because DE is given, the very small Area DLME will also be in the inverse Proportion of that streight Line: Then will the Time be every where propor-

proportionable to the very small Area DLME, and the Sum of all the Times will be proportionable to the Sum of all the Areas (*by Corol. Lem. IV.*) that is, the whole Time which the falling Body takes up in describing the streight Line AE, will be represented by the whole Area ATVME. W.W.D.

Corol. 1. IF a Body be urged by any uniform known centripetal Force (such as Gravity is generally supposed to be, though it really decreases as the Square of the Distance from the Center of Gravity;) and P be the place from whence such a Body must fall, in order to acquire a Velocity in the place D equal to the Velocity acquired in the same place D, by another falling Body, urged by any sort of centripetal Force whatsoever: Then, if in the Perpendicular DF, we take DR, in the same Proportion to DF; that the uniform Force of Gravity bears to the other centripetal Force, and the Rectangle PDRQ be compleated; and if after this we take the Area ABFD equal to this Rectangle, A will be the place from when this other Body fell. Let the Rectangle DRSE be compleated; it was proved above that

the Area ABFD: the Area DFGE:: VV: 2 VI.

or :: $\frac{1}{2}$ V: I. (*dividing by 2 V.*)

that is, the Area ABFD, is to the Area DFGE; as half the whole Velocity acquired by the Body in falling from A to E by the unequal

Q₄

centripe-

centripetal Force; is to the Increment of that Velocity in the place E. By the same way of arguing, the Area PQRD, is to the Area DRSE; as half the whole Velocity acquired by the Body in falling from P to E by the equal uniform Force of Gravity; is to the Increment of the same Velocity in the place E; that is, putting U for this whole Velocity, and J for its Increment.

the Area PQRD: the Area DRSE :: $\frac{1}{2}$ U: J.
But the Times in which these Increments I, J, are generated, being equal, the Increments are as the Forces by which they are generated, that is, as the Ordinates DF, DR (*by the Hypothesis*;) that is,

$$I: J :: DF: DR,$$

consequently I: J :: Area DFGE: Area DRSE,
(because these small Areas are both of the same breadth DE.)

Wherefore
the Area ABFD: the Area DFGE :: $\frac{1}{2}$ V: the Area DFGE.
and

the Area PQRD: the Area DRSE :: $\frac{1}{2}$ U: the Area DRSE (*by substituting for*

for I and J,
the Areas
DFGE
and DRSE.)
Consequent-
ly,

the Area ABFD is proportionable to $\frac{1}{2} V$,
and the Area PQRD is proportionable to $\frac{1}{2} U$
(by Equality of Proportion;) but $\frac{1}{2} V$ is equal
to $\frac{1}{2} U$ (by the Hypothesis;) therefore the Area
ABFD, is equal to the Area PQRD.

Corol. 3. HENCE it follows, that if any
Body be thrown upwards or downwards with a
given Velocity, from any place whatsoever,
suppose from D; and the Law of the centri-
petal Force be given, its Velocity in any other
place, as in *e*, may be found by erecting the
Ordinate *eg*, and taking this Velocity in the
same Proportion to the Velocity it had in the
place D, that a Line equal to the square Root
of the Rectangle PQRD increased by the cur-
vilinear Area, if the place *e* be below the place
D, or diminished by the same Area if *e* be
above D; bears to a streight Line equal to the
square Root of the Rectangle PQRD alone,
that is, putting U for the Velocity in D, and
V for the Velocity in *e*, it will be,

$$U:V::\sqrt{PQRD}:\sqrt{PQRD+DFGE}.$$

Corol. 3. THE Time may also be found by
erecting the Ordinate *em* reciprocally propor-
tionable to the square Root of the Area
PQRD, increased or diminished by the Area
DFGE; and taking the Time which the Body
takes

takes up in falling through the Space De , to the Time which another Body, urged by the uniform Force of Gravity, takes up in falling from P to D ; as the curvilinear Area $DLme$, to the Rectangle $2PD \times DL$. For the Time which a Body, descending by the uniform Force of Gravity, takes up in falling through the Space PD ; is to the Time which the same Body takes up in falling from P to E , as the square Root of PD , to the square Root of PE ; that is, putting T and T for those Times.

$$T : T :: \sqrt{PD} : \sqrt{PE}. \text{ (by Scholion Corol. 6. of the Third Law of Motion.)}$$

that is, $T : T :: PD : PD + \frac{1}{2} DE$. (the Line DE being supposed infinitely small, or beginning to be generated :) For suppose the Space PD to be represented by the Square $ABCD$, and the Space PE to be represented by the Square $AGFE$; then will DE , the difference of these Spaces, be represented by the Rectangle $DCBGFE$, which is the difference of the Squares; for the same Reason AD will be as the Root of PD , and AE as the Root of PE ; but AD is to AE as the Square $ABCD$ to the Square $ABCD + \frac{1}{2} DCBGFE$ (by the Elements;) therefore $\sqrt{PD} : \sqrt{PE} :: PD : PD + \frac{1}{2} DE$. (by Substitution.)

$$\text{or } :: 2PD : 2PD + DE. \text{ (multiplying by 2.)}$$

$$\text{And, } T : T - T :: 2PD : DE. \text{ (by Division.)}$$

tion;) that is, the Time which the Body takes up in falling through the Space PD, by the uniform Force of Gravity, is to the Time which the same Body takes up in falling through the very small Space DE; as $2PD$ to DE , or as the Rectangle $2PD \times DL$, to the Area $DLME$. (*by multiplying these last Terms by DL ;*) that is,

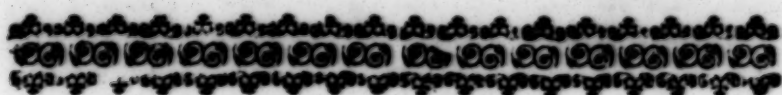
$T : T---T :: 2PD \times DL : \text{the Area } DLME$.

Further, the Time in which each of the Bodies describe the small Line DE , is to the Time in which the Body, which is moved by the unequal centripetal Force falls through the Space De , as the Area $DLME$, to the Area $DLme$, (*by the Hypothesis.*) that is, putting $\mathcal{T}$ for the Time of the Body's falling through the Space De .

$T---T : \mathcal{T} :: \text{the Area } DLME : \text{the Area } DLme$.

Wherefore $T : \mathcal{T} :: 2PD \times PL : DLme$ (*by Equality of Proportion.*) that is, the Time of a Body's falling thro' the Space PD by the uniform Force of Gravity, is to the Time in which another Body falls through the Space De , by any unequal centripetal Force whatsoever; as the Rectangle $2PD \times DL$, to the Area $DLme$.

SECTION



SECTION X.

Concerning the Motion of Bodies vibrating in Pendulums.

PROPOSITION XLVIII. THEOREM XVI.

If a Wheel be placed at right Angles to the external Surface of a Globe, and revolves after the manner of a Wheel, in any great Circle of that Globe; the length of the curvilinear Path which any given Point, in the Circumference of the Wheel which touches the Globe, describes; (which curvilinear Path may be called a Cycloid or an Epicycloid) will be to double the versed Sine of half the Arch which has touched the Globe, during the Time that the Wheel has moved along; as the Sum of the Diameters of the Globe and the Wheel, to Semidiameter of the Globe, that is,

$$AP : BV---VP :: 2CE : CB.$$

PROPOSITION XLIX. THEOREM XVII.

If a Wheel be placed at right Angles on the internal concave side of a Globe, and revolves

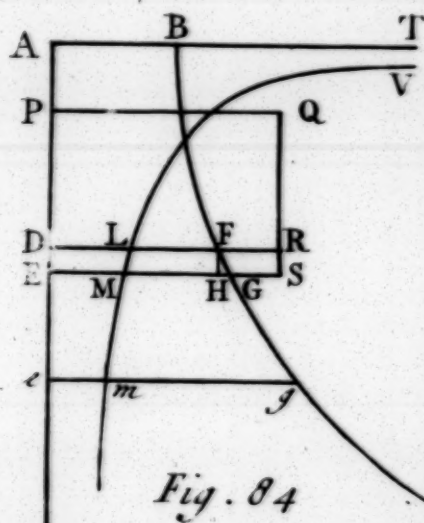
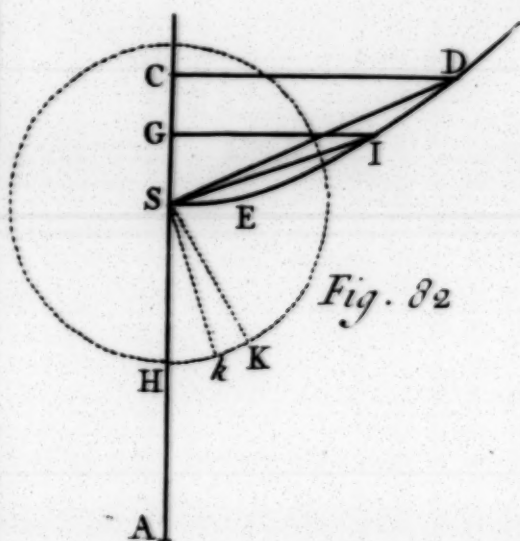
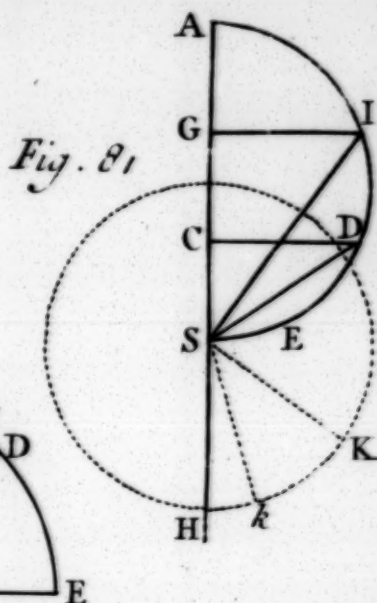
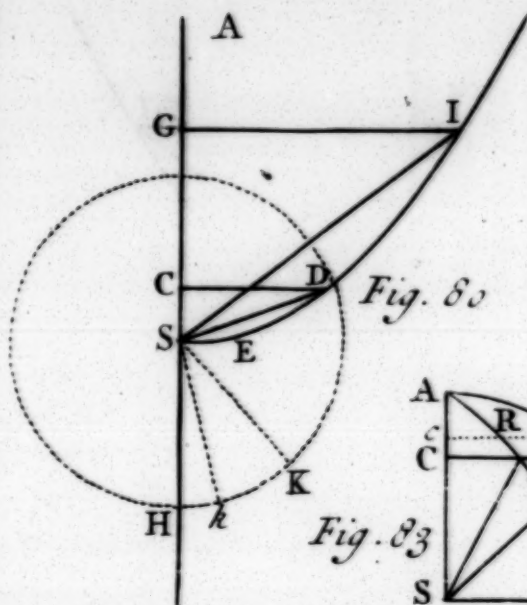
in

I.

x-
er
of
ar
n-
be,
be
to
cb
at
of
to

II.

in-
ves
in



J
t
i
t
t
f
A
h
c
t
v
t
h
c

in any great Circle; the length of the curvilinear Path which any given Point in the Circumference of the Wheel which touches the Globe, describes; will be to double the versed Sine of half the Arch which has touched the Globe during the Time that the Wheel has moved along; as the difference of the Diameters of the Globe and Wheel, to the Semi- Fig. 196.
diameter of the Globe, that is;

$$AP : BV - VP :: 2CB - 2BE : CB.$$

LET ABL represent a Globe, C its Center, BPV a Wheel placed upon it, E the Center of the Wheel, B the Point where it touches the Globe, and P the given Point in the Circumference of the Wheel. Suppose this Wheel to go along in the great Circle ABL from A, through B to L, and as it goes, to be so turned about its Center E, that the Arches AB, PB may be always equal; and that this point P which is given in the Circumference of the Wheel, may at the same time describe the curvilinear Path AP. Now let AP be the whole curvilinear Path described from the Time that the Wheel touches the Globe in A; and the length of this curvilinear Path AP, will be to double the versed Sine of the Arch $\frac{1}{2}$ PB; as $2CE$, to CB . For let CE (produced if need be) meet the Wheel in V , and let CP, BP, EP, VP , be joined, and let VF be let fall perpendicular upon CP produced. Let PH, VH touch the Wheel in P and V , and meet each

each other in H , and PH will intersect VF in G ; then let GI and HK , be let fall perpendicular to VP . On the Center C , and with any distance Cn , let the Circle nom be described, crossing the streight Line CP in n , the Circumference of the Wheel in o , and the curve Line AP in m : and on the Center V , and with the distance Vo , let the Circle oq be described, intersecting VP produced in q . Now because the Wheel, as it goes along, is always turned about the Point of Contact B , it is manifest that the streight Line BP (which is the Radius of the Curvature of the Curve AP in the point P ,) is perpendicular to the curve Line AP , described by the point P of the Wheel, and that therefore the streight Line VP touches this Curve in the point P ; for BPV is a right Angle, because it is contained in the Semicircle BPV , (*by the Elements.*) and therefore VP is perpendicular to BP . If Cn , the Radius of the Circle nom , be gradually increased or decreased, it will at length become equal to CP . Let the point n be infinitely near the point P ; then if the Arch nom be described, it is manifest, that in the same time that the curve Line Am is increased by the very small Increment mP , the streight Line Cn is increased by the very small Increment nP . For the same Reason, if the small Arch oq be described with the Radius Vo , the Increment Po in the Circle BPV , will be generated in the same Moment of Time that Pq is generated in the streight Line VP , and that the fore-mentioned

tioned Increments were generated in. Now these incremental Arches Po , Pm , may be considered as streight Lines (*by Lem. VII. Corol. 2, 3.*) and then, because the Angles at n and F are right Angles, and the Angle $n P m$ equal to the Angle $V P F$, therefore the Triangles $n P m$, $V P F$ are similar, consequently

$$Pm : Pn :: PV : PF.$$

So likewise, because the Angles at q and I are right Angles, and the Angle $o P q$ equal to the Angle $I P G$; therefore the Triangles $o P q$ and $I P G$ are similar to each other, and to the former Triangles, (*by Prop. VIII. Book VI. of the Elements.*) consequently,

$$Po : Pq :: PG : PI. \text{ that is,}$$

the last Proportion of the vanishing Lines Pm , Pn , Po , Pq , or the Proportion of the instantaneous Variations of the curve AP , the streight Line CP , the circular Arch BP , and the streight Line VP , is the same as the Proportion of the Lines PV , PF , PG , PI , to each other. Now because VF is perpendicular to CF , and VH perpendicular to CH , the Angle HVG is therefore equal to the Angle VCF (*by Prop. VIII. Book VI. of the Elements.*) and the Angle VHG is equal to the Angle CEP , because in the quadrilateral Figure $HVEP$, the Angles at V and P being right Angles, the other two Angles VEP , VHP together, must be equal to two right Angles; the Angles VEP and CEP are also equal to two right Angles; therefore if we take away the common Angle VEP , there will remain VHG or VHP , equal to CEP ;
therefore

therefore the Triangles VHG and CEP are similar, and consequently;

$EP : CE :: HG : HV$ or HP . because VH and HP are Tangents to the points of the Circle V, P , and therefore equal to each other: And because the Triangles HKP, GIP , are similar;

$$HG : HP :: KI : KP.$$

And $EP : CE :: KI : KP$. (*by Equality of Proportion.*)

consequently $CB (= CE + EP) : CE :: PI (= KP + KI) : KP$. (*by Composition and Division.*)

And $CB : 2CE :: PI : 2KP = PV$. (*by doubling the Consequents.*)

But $PI : PV :: Pq : Pm$ (*as was proved before.*)

There-fore $CB : 2CE :: Pq : Pm$. (*by Equality of Proportion.*) that is,

Pq the Decrement of the Line VP , or which is the same thing, the Increment of $BV - VP (= PZ$, in the following Figure.) is to Pm , the Increment of the curve Line AP generated in the same Moment of Time; in the given or constant Proportion of CB to $2CE$; now as this Proportion holds true in every point of the Curve AP , and in every point of the straight

streight Line $BV - VP$, or PZ , and as this Curve and this streight Line began to be generated together, and continue to be so; therefore (by *Corol. Lem. IV.*) all the Increments of which the curve Line AP is composed, and which are equal in number and correspondent to all the Increments of which the streight Line $BV - VP$ or PZ , is composed; that is, the length of the Curve AP , is to the length of the Line $BV - VP$ or PZ , generated by these Increments, in the same Proportion of $2CE$, to CB . Now if BV be the Radius, VP will be the Co-sine of the Angle PVP , or of half the Angle BEP (by *Prop. XX. Book III. of the Elements.*) and therefore BV or $ZV - VP$, that is, PZ , is the versed Sine of the same Angle; wherefore in this Wheel, whose Radius is $\frac{1}{2}BV$; $BV - VP$ is double the versed Sine of the Arch $\frac{1}{2}BP$. Whence it follows, that AP is to double the versed Sine of the Arch $\frac{1}{2}BP$, as $2CE$ to CB . W. W. D.

Fig. 197.

THE Line AP in the first Proposition, may be called the Cycloid on the outside of the Globe, and the Line AP in the latter Proposition may be called the Cycloid on the inside of the Globe, in order to distinguish them from each other.

Corol. I. HENCE it follows, that if the whole Cycloid ASL be described and bisected in S ; the length of the part PS , will be to the Line VP (which is double the Sine of the Angle VBP , if EB be made Radius; for EZ is the

R

Sine

Fig. 198.

Sine of the Angle VBP to the Radius EB , and VP is equal to $2EZ$, *by the Elements.*) as $2CE$ to CB , and therefore PS and VP are in a given Proportion to each other.

Corol. 2. AND the length of AS , which is half the Circumference of the Cycloid, is equal to a streight Line, which bears the same Proportion to the Diameter of the Wheel BV , that $2CE$ bears to CB ; for when the Wheel arrives in the point S , the points P, V and S will coincide, and $BV - VP$ will become equal to BV . consequently

$$AS : BV :: 2CE : CB.$$

PROPOSITION L. PROBLEM XXXIII.

How to make a Pendulum vibrate in a given Cycloid.

Fig. 199. LET the given Cycloid QRS , be described within the Globe QVS , whose Center is C ; and let this Cycloid be bisected in R , and let the extreme Points of it meet the Superficies of the Globe in Q and S . Let CR be drawn bisecting the Arch QS in O , and let it be produced to A , so that CA may bear the same Proportion to CO , that CO bears to CR . Then on the Center C , and with the Distance CA , let the external Globe DAF be described; and within this Globe, and by a Wheel whose Diameter is AO , let there be described two Semicycloids AQ, AS , which may touch the
internal

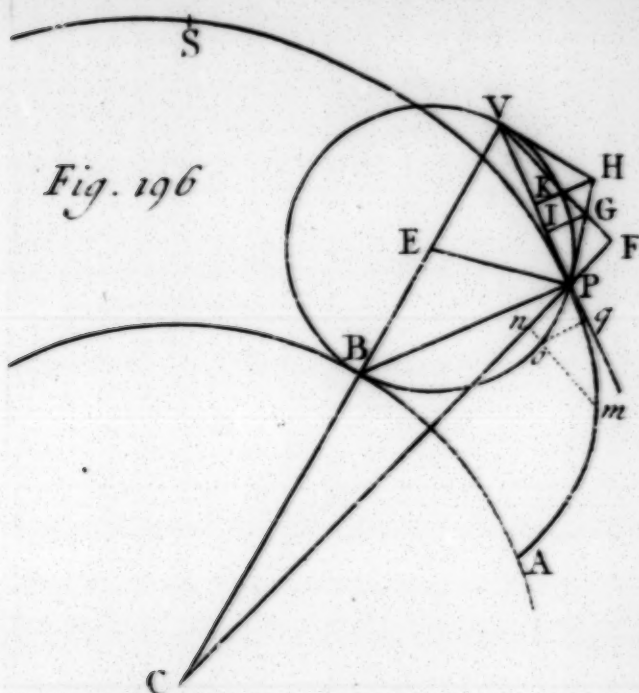
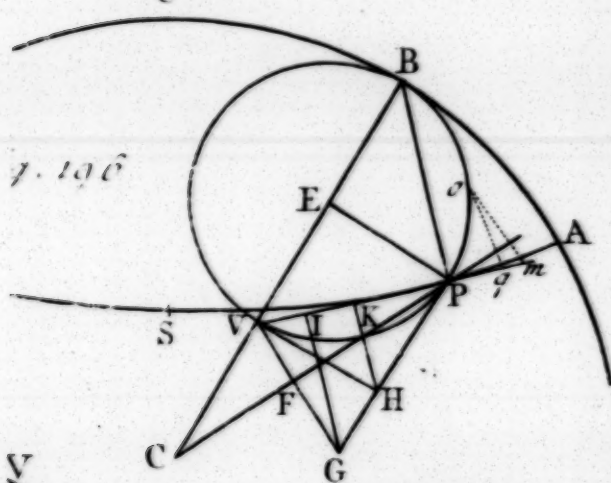


Fig. 196



7. 196

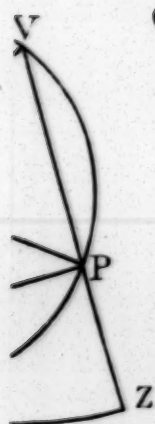


Fig. 198

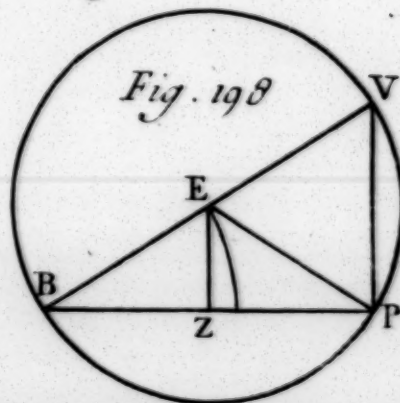
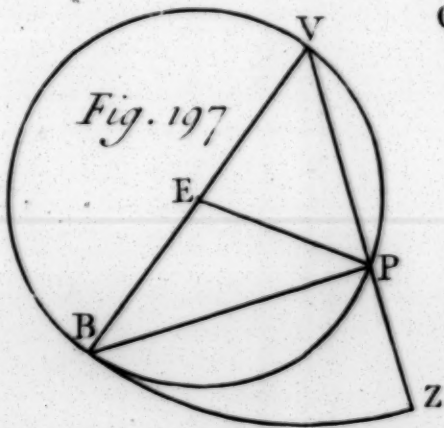
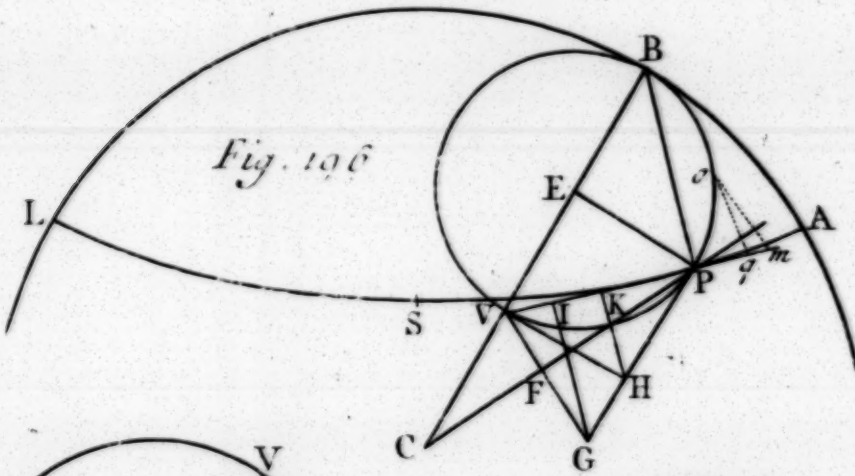
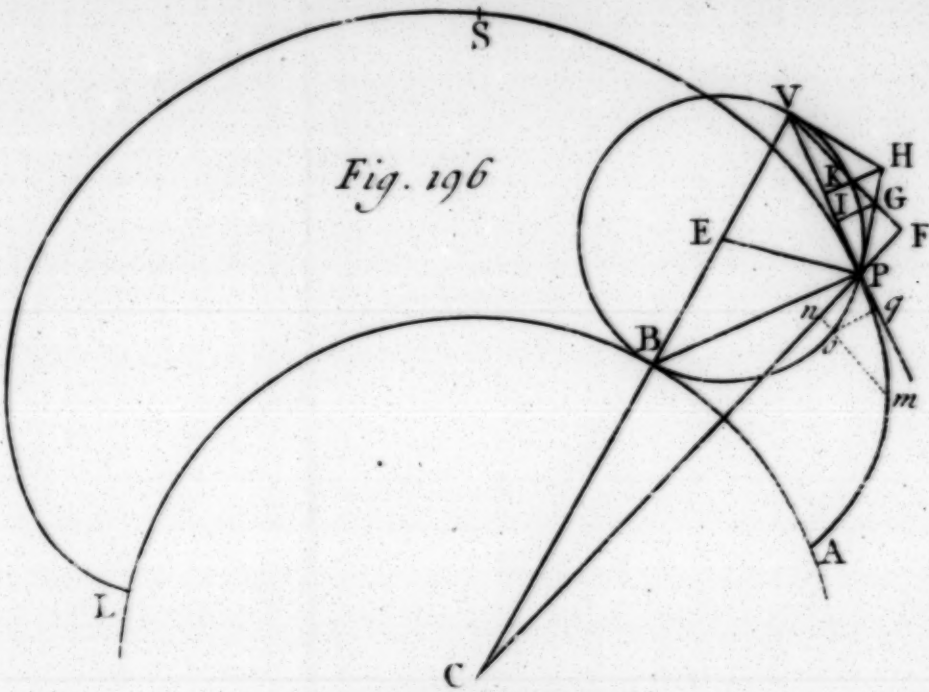
h.
(
P

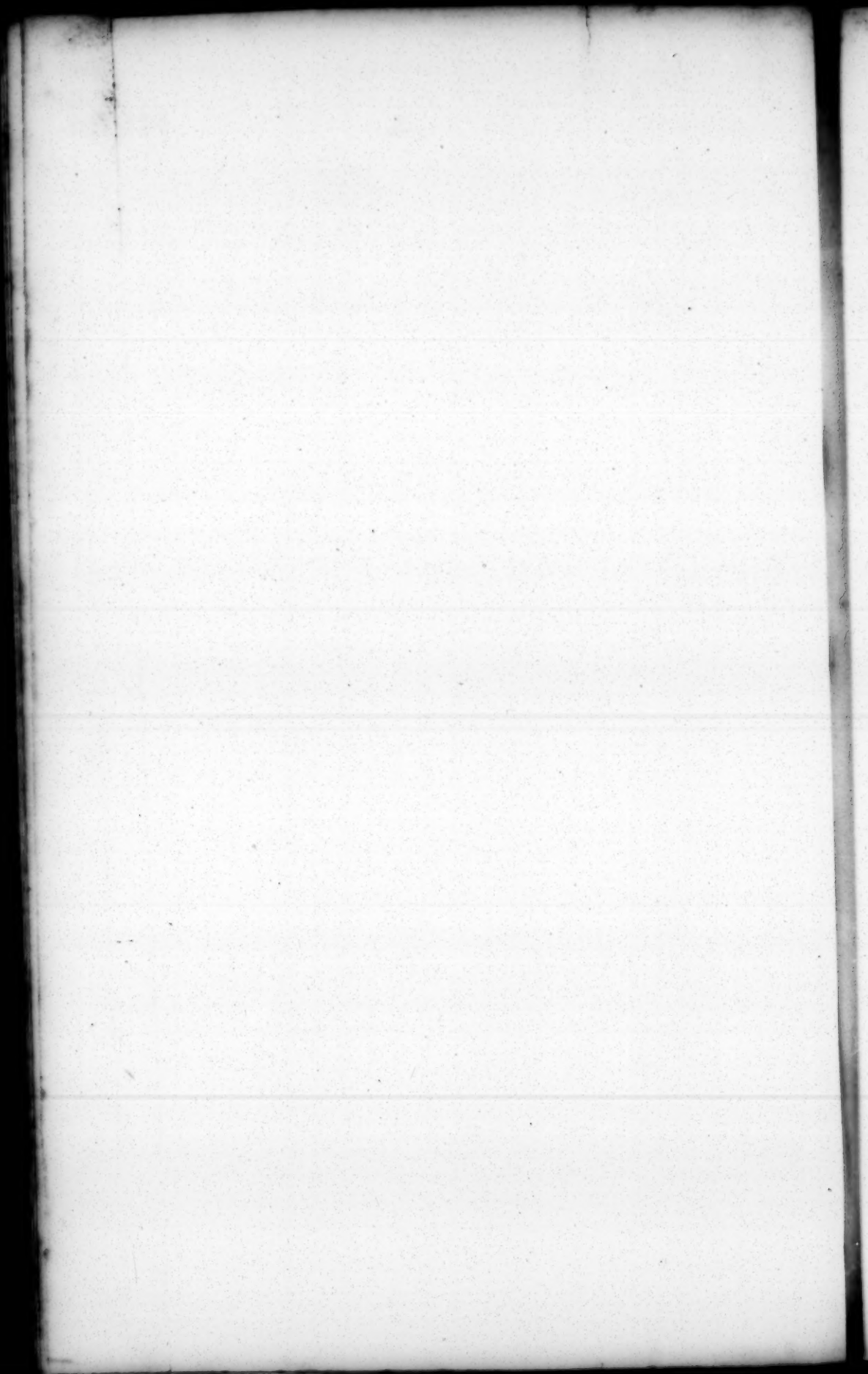
is
al
o-
y,
el
S
ne

I.

oen

bed
C;
let
s of
bi-
pro-
ame
C.R.
nce
bed;
nose
two
the
ernal





internal Globe in Q and S , and meet the external Globe in A . Let the Body T be suspended on the point A , by a Thread APT , whose Length is equal to AR , and let it vibrate between the Semicycloids AQ , AS , in such a manner, that as often as the Pendulum goes from the Perpendicular AR , the upper-part of the Thread AP may be applied to that Semicycloid APS , towards which the Body T moves, and be bent about it, as being in its way; and let the remaining part PT , which has not yet met the Semicycloid, be stretched out into a streight Line; then will the Weight T vibrate in the given Cycloid QRS . $W.W.S.$

FOR let the Thread PT meet the Cycloid QRS in T , and the Circle QOS in V , and let the Line CV be drawn; and on the extreme points P and T of the streight part of the Thread, let the Perpendiculars BP , TW , be erected, meeting the streight Line CV produced, in B and W . Then because the generating Wheels FQ and QX are to each other in the given Proportion of AO to OR , and the Arches of the directing Circles AF and SQ , are fimilar, and both the Wheels begin to generate the two Semicycloids QA , QR , in the same point Q ; it is manifest, from the Construction and Generation of these two Figures, that they must be fimilar to each other, and the elementary parts of each Curve generated in the same or equal parts of Time, proportionable to each other; and the same

Thing holds true for the other two Semicycloids AS, SR; wherefore the two Perpendiculars PB, TW, must cut off from the Line CV, the Lengths VB, VW, equal to the Diameters of the Wheels OA, OR (*by Prop. XLIX.*) Wherefore TP, is to VP, (which is double the Sine of the Angle VBP, when $\frac{1}{2}$ BV is made Radius.) as BW, to BV. for the Triangles VBP, VTW, being right-angled at P and T, and the Angle TVW equal to the Angle BVP, are similar;

consequently. $VT : VP :: VW : BV.$
 And (*by Composition.*) $VT + VP = TP : VP :: VW + VB = BW : BV.$

or $TP : VP :: AO + OR : AO.$
 Then because (*by the Hypothesis,*) $CA : CO :: CO : CR.$
 and (*by Division.*) $CA : CO :: CA - CO = AO : CO - CR = OR.$

Therefore $TP : VP :: CA + CO : CA.$
 by substituting in the Proportion $AO + OR : AO$; CA instead of AO, and CO instead of OR. Or if BV be bisected in E, it will be
 $TP : VP :: 2CE : CB.$

Whence it follows (*by Cor. 1. Prop. XLIX.*) that the length of the streight part of the Thread PT is every where equal to the Arch of the Cycloid PS; and that the whole Thread APT, is always equal to APS, the Arch of the half Cycloid APS; that is, to the Line AR (*by Corol. 2. Prop. XLIX.*) Wherefore, on the contrary, if the Thread continues always equal to the Line AR, the point T will be moved in the given Cycloid QRS. W.W.D.

Corol.

Corol. The Thread AR is equal to the Semicycloid AS, and therefore it bears the same Proportion to AC, the Semidiameter of the external Globe; that the Semicycloid SR, which is similar to it, bears to CO the Semidiameter of the internal Globe; that is,

$$AR = AS : AC :: SR : CO.$$

because the homologous or correspondent parts or sides of all similar Figures, are proportionable to each other.

PROPOSITION LI. THEOREM XVIII.

If the centripetal Force that tends from all parts to the Center C, be in every place, in a direct Proportion of the Distance of that place from the Center, and the Body T vibrates backward and forward by that Force only acting upon it (in the manner just now described) in the Circumference of the Cycloid QRS; I say, that let the Vibrations be never so unequal, they will all be performed in the same Time.

FOR let any Tangent TW be infinitely produced, and CX be let fall perpendicular to it, and let CT be joined. Then, because the centripetal Force by which the Body is impelled towards C, is proportionable to the Distance CT, (*by the Hypothesis.*) and this Force may be resolved into two parts, proportionable to the Lines CX, TX, (*by Corol. 2. of the Laws*

Fig. 200.

of Motion.) one of which, *viz.* CX, because it impells the Body according to the Direction of the Thread PT, is spent in stretching that Thread, and being overcome by the Resistance of it, is wholly lost, and can produce no other Effect; but the Force proportionable to the other Line TX, because it acts upon the Body T transversely, or according to the Direction TX, must accelerate its Motion directly in the Cycloid: Therefore it is manifest, that the Acceleration of the Body, which is proportionable to this accelerative Force, must be every Moment proportionable to the Line TX, or to the length of the Arch of the Cycloid TR. For the Triangles VTW, WXC being rectangular at T and X, and the Angles CWX, TWV, equal; are similar; therefore

$$\begin{array}{l} WC \quad \quad \quad ; WV :: WX \quad \quad \quad : TW. \\ \text{and } WC + WV = CV : WV :: WX + WT = TX : TW. \end{array}$$

(by Composition.)

Therefore $CV \times TW = WV \times TX$, *(by multiplying the Extremes and Means.)* and because CV and WV are given Quantities, therefore TW is proportionable to TX; now TW is proportionable to the length of the Arch of the Cycloid TR, *(by Corol. 2. Prop. XLIX.)* Wherefore the accelerative Force is also proportionable to the same Arch TR. Wherefore, if two Pendulums, as APT, Apt, be removed from the Perpendicular AR, to any unequal Distances TR, tR, and then be let fall both together

together, they will be accelerated in every point as they are falling, exactly in Proportion to the Arches which they are to describe, that is, in Proportion to the Arches TR , tR . Now the infinitely small parts of the Cycloid, which are described by these Pendulums, in the first Moment of Time, when they are let fall from the places T , and t ; are proportionable to their Accelerations, or proportionable to the whole Arches, which they are to describe; and consequently, (*by Division.*) the parts which remain to be described, and the succeeding Accelerations which are proportionable to those parts, are also proportionable to the whole Arches. Thus suppose the Pendulum APT to fall through tr , a very small part of the Arch SR , in the first Moment of Time; and the Pendulum Apt , to fall through Tu , another very small part of the same Arch, in the same Time; then because the Accelerations with which these small parts tr , Tu , are described, are proportionable to the whole Arches tR , TR ; therefore the parts themselves tr , Tu , described by these Accelerations, are proportionable to the whole Arches tR , TR ; and for the same Reason, the next very small parts to be described by these Pendulums in falling from r and u , will be proportionable to the Arches rR and uR ; and so on till the Pendulums arrive at the lowest point R . The Accelerations therefore, and the Velocities acquired, and the small parts tr , Tu , described

by these Velocities, and all the parts to be described afterwards, are every where as the whole Arches, that is, as tR , TR , rR , uR , &c. and consequently, the parts to be described, being always in the same Proportion to each other, must vanish together, or when one of them becomes equal to nothing, the other which is always in the same Proportion with it, will become equal to nothing also: for suppose the Parts which are to be described, to be to each other in the given Proportion of a to b , or $tR : TR :: a : b$; then if T coincides with R , it will be, $tR : o :: a : b$ or $tR = \frac{o a}{b} = o$. that is, tR will also become equal to nothing at the same time; or the two vibrating Bodies will both arrive at the Perpendicular AR at the same time. And because, on the other hand, the Ascents of the Pendulums from the lowest place R , through the same Arches of the Cycloid the contrary way to the former, are retarded by the same Forces with which their Descents were accelerated; it is manifest, that the Velocities with which they ascend and descend through the same Arches are equal, and are therefore performed in equal times; and consequently when RS and RQ , two parts of the Cycloid, lying on each side of the Perpendicular AR , are similar and equal; then the two Pendulums will always perform their whole as well as half their Vibrations in the same times. W. W. D.

Corol.

Corol. THE Force with which the Body T is accelerated or retarded in any place of the Cycloid T, is to the whole Weight of the same Body in the highest place S or Q; as the Arch of the Cycloid TR, to the Arch SR or QR of the same Cycloid. For when the Body T is in S, the Lines TC and TX will coincide, and consequently the Line CX, which represents that part of the Force of Gravitation which is employed in stretching the Thread of the Pendulum, vanishes; and therefore the Arch SR must represent the whole Gravitation.

PROPOSITION LII. PROBLEM XXXIV.

How to determine both the Velocities of Pendulums in every place of their Vibrations, and the Times in which their whole Vibrations, or any particular part of their Vibrations are performed.

ON any Center G, with the Distance GH, Fig. 201, equal to RS the Arch of a Cycloid, let the Semicircle HKM be described, and bisected by the Semidiameter GK. And if the centripetal Force tends to the Center G, and be every where proportionable to the Distances of the several places from this Center; and if this Force in the Circumference HIK, be equal to the centripetal Force on the Surface of the Globe QOS tending to its Center C; and if at the same Moment of Time that the Pendulum T is let fall from the highest place S, a
Body

Body such as L begins to fall by its own Gravity from H towards G: then because the Forces with which the pendulous Body and this other Body are urged, are equal at the beginning of their Motions, and are every where proportionable to the equal Spaces TR and LG, described by them; and therefore if TR and LG be equal, these Forces will be equal also; it is manifest, that the Bodies T and L do describe equal Spaces ST, and HL, at the beginning, and consequently that they will go on to be urged equally by their respective Forces, and to describe equal Spaces afterwards. Now (*by Prop. XXXVIII.*) the Times which the Body L takes up in falling from H to L, or from H to G, are proportionable to the Arches of the Circumference HI and HK; and therefore the Time in which the pendulous Body describes the Arch ST, which is equal to LH, (*by the Hypothesis*) is to the Time of one entire Vibration, *viz.* from S to Q, through the Arch SRQ, equal to MH; as the Arch HI, which represents the Time in which the Body L would fall from H to L, to half the Circumference HKM, which represents the Time which the Body L would take up in falling through the Diameter from H to M. And because (*by the same Proposition, viz. XXXVIII.*) the Velocities of the falling Body L are every where proportionable to the Sines drawn from the places where the Body is, that is, the Velocities in L and G, are as the Lines LI and GK;

GK; therefore the Velocity of the pendulous Body in T, is to its Velocity in the lowest place R; (that is, the Velocity of the Body L in the place L, is to its Velocity in the place G, or supposing the Arches HI, HK, to be generated by an equable Motion, and consequently to increase equally; the momentaneous Increment of the Line HL, is to the momentaneous Increment of the Line HG, *See the Fig. to Prop. 38.*) as the ordinate Applicate or right Sine IL, to the Radius GK. or as $\sqrt{SR^2 - TR^2}$ to SR. (by Prop. XLVII. Book I. of the Elements, and because SR is equal to GI or the Radius, and TR is equal to GL.) Hitherto we suppose the Vibrations of the Pendulum to be all equal, or that the vibrating Body moves backward and forward from S to Q, and from Q to S; if we suppose further that the Vibrations are unequal, and that the Pendulum is retarded, so that instead of moving from S to Q, it moves from T to W, (WR being taken equal to TR;) then because the whole Vibrations from S to Q, and from T to W, are performed in the same Time (by the last Prop.) if this Time be divided into an equal Number of very small Parts, the correspondent Arches described in each of the Vibrations in any equal Number of these small parts of the Time, will be proportionable to the whole Arches described by the whole Vibrations; (by the last Prop.) that is, if the Pendulum moves from T to V in the same Time that it moves from S to T, the
 Arches

Arches TV , ST , will be to each other as the Arches TW , SQ ; consequently, if the Times be given, the Velocities and the Arches described are had in all Vibrations whatsoever equal or unequal. Thus suppose the Pendulum AT to be let fall from the Point T , and it were required to find the Arch TV described in a given Time, and the Velocity at V acquired in that Time. Let another Pendulum of the same Length with AT , be let fall from the Point S , at the same time that the Pendulum AT is let fall from T ; then by what has been already said, if we take TV in the same Proportion to TW that ST bears to SQ ; V will be the place to which the Pendulum AT will fall in the time that the Pendulum AS fell from S to T , consequently TV is the Arch required, and is also proportionable to the Velocity acquired at V ; which were the first things to be found.

Fig. 201, 202. LET us now suppose two Pendulums AT , at , vibrating in the two different Cycloids SRQ , srq , and within the two different Globes SOQ , soq ; and let the whole Force of the Globe QOS be called F , and the whole Force of the Globe qos be called f . Then (*by the Hypothesis.*) the accelerative Force, with which the Pendulum AT is urged when the Body T is in the Point S , in the Circumference of the Globe SOQ , where it begins to be moved directly towards the Center C ; will be as the Distance of this Body from the Center; and

and the whole Force jointly, or as $CO \times F$. For the same Reason the accelerative Force of the other Pendulum at , in the point f of the Circumference of the other Globe, will be as $co \times f$. Then if the small Spaces HY , by , be described in equal small parts of Time, they will be as the accelerative Forces $CO \times F$, $co \times f$, (by Definition VII.) and if from the Points Y , y , the Perpendiculars YZ , yz , be erected, meeting the Circumferences in Z , z , the infinitely small Arches HZ , bz , will represent those equal small parts of Time (by Prop. XXXVIII.) Now these very small Arches HZ , bz , are (by the Elementes.) in a subduplicate Proportion of the Rectangles $GH \times HY$, and $gb \times by$; that is, as $\sqrt{GH \times CO \times F}$, and $\sqrt{gb \times co \times f}$, (because HY and by are proportionable to $CO \times F$ and $co \times f$.) Whence it follows, that the Time of one entire Vibration in the Cycloid QRS , is to the Time of one entire Vibration in the Cycloid qrs , in a Proportion compounded of GH to gb directly, and $\sqrt{GH \times CO \times F}$, to $\sqrt{gb \times co \times f}$ inversely; that is, putting T for the Time of the whole Vibration of the Pendulum TA , and t for the Time of the whole Vibration of the Pendulum ta , it will be

$$T : t :: \frac{GH}{\sqrt{GH \times CO \times F}} : \frac{gb}{\sqrt{gb \times co \times f}}.$$

(for the Times of the entire Vibrations T , t , are as the half Circumferences HKM , $bk m$, which represent those entire Vibrations directly;

Concerning the Motion of BODIES

and the small Arches HZ, bz, which also represent the small given Particles of Time inversely :) that is,

$$T : t :: \frac{HKM}{HZ} : \frac{bkm}{bz}.$$

for let P represent the given Particle of Time in which the Body moves from H to Y, or from b to y; then (by Prop. XXXVIII.)

$$T : P :: HKM : HZ.$$

$$\text{and } P : t :: bz : bkm.$$

therefore $T : t :: \frac{HKM}{HZ} : \frac{bkm}{bz}$ (by Composition fore

but HKM and bkm are as GH, gb, (by the Elements.) and HZ, bz were before shown to be as $\sqrt{GH \times CO \times F}$, and $\sqrt{gb \times co \times f}$, therefore;

$$T : t :: \sqrt{\frac{GH}{GH \times CO \times F}} : \sqrt{\frac{gb}{gb \times co \times f}} \quad (\text{by Substitution.})$$

$$\text{or } T : t :: \sqrt{\frac{SR}{CO \times F}} : \sqrt{\frac{sr}{co \times f}} \quad (\text{by Substitution and Division, because } SR = GH, \text{ and } sr = gb \text{ by the Hypothesis.})$$

$$\text{or } T : t :: \sqrt{\frac{AR}{AC \times F}} : \sqrt{\frac{ar}{ac \times f}} \quad (\text{for by Cor.}$$

Prop.

Prop. L. SR
is to CO as
AR to AC.)

Wherefore the whole Vibrations in all sorts of Globes and Cycloids, made with any Forces whatsoever, are in a Proportion compounded of the subduplicate Proportion of the Length of the Threads directly, and a subduplicate Proportion of the Distance betwixt the Point of Suspension and the Center of the Globe inversely, and a subduplicate Proportion of the absolute Force of the Globe inversely also. W. W. S.

Corol. I. HENCE we may find how to compare the Times of vibrating, falling and revolving Bodies with each other. For if VB the Diameter of the Wheel BPV, (*See the Fig. to Prop. XLIX.*) by which the Cycloid LSA is described within the Globe, be made equal to CB, the Semidiameter of the Globe, then the Cycloid LSA will become a straight Line passing through C, the Center of the Globe; and then the Vibration will become a direct Descent down to this Center, and after that a direct Ascent from it upwards. Wherefore since these Descents from any places whatsoever, as from H, Y, L to the Center G, *Fig. 201;* are performed in the same Time, and this *202.*

Time is equal to the Time that a Body revolving uniformly about the Center of the Globe G, at any of the Distances HG, YG, and LG, would describe the Arch of a Quadrant; therefore the Time of describing such

an

an Arch is given ; for it is (*by Case the Second.*) to the Time of half a Vibration in any Cycloid

QRS, as 1, to $\sqrt{\frac{AR}{AC}}$. Thus suppose it were

required to find the periodical Time of a Body revolving about the Center of the Earth in the Circumference of it, or at the Distance of the Semidiameter ; It appears by Experiments, that a Pendulum of 3,058 *Paris* Feet, takes up exactly a Second of Time every Vibration, and the Semidiameter of the Earth is computed to be 19615800 *Paris* Feet, wherefore (*by this Corol.*) the Time of performing a fourth Part of its Revolution, is to the Time of half a Vibration, that is, to half a Second ; as 1 is to

$\sqrt{\frac{3,058}{19615803,058}}$, or as 1 to $\frac{1,75}{4429}$. (*See Prop.*

XXXVI.) that is, about twenty one Minutes ; so that a Body would descend from the Superficies of the Earth, or perform one fourth part of a Revolution about its Center at that Distance, in about twenty one Minutes, or it would describe one whole Revolution, in about eighty four Minutes ; which may be confirmed by computing the same thing according to *Corol. 6. Prop. IV.* where it was proved that when Bodies revolve about a common Center, the Squares of their periodical Times are as the Cubes of their Distances ; so that if we take the Moon's Distance from the Center of the Earth, and her periodical Time, and the Earth's Semidiameter ; the periodical Time of
a Bodys

a Body revolving about the Earth's Center at the Distance of its Semidiameter; will come out exactly the same, viz. 84 Minutes. (See Page 129.)

Corol. THE same things also follow from hence, which were found out by Sir *Christopher Wren* and Mr. *Hugens*, concerning the common Cycloid. For if the Diameter of the Globe CB (See *Fig. to Prop. XLIX.*) be increased infinitely, then the spherical Superficies will become a Plane, and the centripetal Force will act uniformly in the Direction of Lines perpendicular to this Plane; and the Cycloid in this Figure will become the common Cycloid described by a Wheel upon a Plane. And in this Case, the length of the Arch of the Cycloid between this Plane and the describing Point, will become equal to four times the versed Sine of half that Arch of the Wheel which is between the Plane and the describing Point; as was found out by Sir *Christopher Wren*. And a Pendulum will vibrate between two such Cycloids, in a Cycloid which is similar and equal to them, and the Times of its Vibrations will be all equal, as Mr. *Hugens* has demonstrated. And the Descent of heavy Bodies, in the time of one Vibration, will be found likewise to be the same, as is demonstrated by Mr. *Hugens*. (See his *Horologium Oscillatorium*.)

THE foregoing Propositions, which we have demonstrated, are accommodated to the

S

true

true Frame and Constitution of the Earth; because when the Wheels move in great Circles upon the Earth, they describe Cycloids out of the Globe, by Pins fixed on their Circumferences; and Pendulums suspended in Mines or Caverns of the Earth, ought to vibrate in Cycloids which are within the Globes, in order to have all their Vibrations performed in the same Time. For Gravitation (as is shown in the third Book) decreases every way as we go from the Superficies of the Earth upwards in a duplicate Proportion of the several Distances from its Center, but as we go downwards within its Superficies, the Gravitation decreases in a simple Proportion of the several Distances from the Center.

S C H O L I U M.

THE Invention of the Pendulum, is of excellent Use in Astronomy, and where-ever there is occasion to know the true Time; for by this means we have a certain Measure of it, by which we can compute it with the utmost Exactness, by half Seconds or Seconds, or any other parts of Time, according to the different Lengths of the Pendulums; and till this Instrument was found out, there was no certain Measure of Time; for there being no Bodies that we know of, whose Motion is exactly equable, and the parts of Time flowing all equably, whatever moveable Bodies were made

use of, they deviated from the true Measure, proportionably to the Inequality of their Motion, so that by this Instrument all others may be regulated, and made fit for common Use. The Pendulum is also of great Use to determine the true Figure of the Body of the Earth; Because the Earth revolves about its own Axis once in twenty four Hours, and because it is not one entire solid Body, but consists partly of a Fluid, and partly of a great Number of different sorts of Bodies which are not closely united to each other; therefore the Figure of the Earth cannot be an exact Globe, but a Sphæroid, whose Diameter at the æquinoctial Circle is longer than its Axis; for by its revolving about its Axis, all the parts of Matter of which it is composed, except those which lie in the Axis it self, must describe so many concentrick Circles; and because they all describe similar Arches of those Circles in the same Time, therefore their Velocities will be as their nearest Distances from the Axis of the Earth; whence it follows, that some part of the Gravitation of these Bodies of which the Earth is composed, must be taken off by this circular Motion, and that in a certain Proportion to the Swiftness of the Motion; and consequently at the Equinoctial, where the Motion is swiftest of all, the Gravity must be most diminished, the Effect of which must be, that the lighter parts giving way to the heavier, the Earth must be higher at the Equator than at the Poles, where there

true Frame and Constitution of the Earth; because when the Wheels move in great Circles upon the Earth, they describe Cycloids out of the Globe, by Pins fixed on their Circumferences; and Pendulums suspended in Mines or Caverns of the Earth, ought to vibrate in Cycloids which are within the Globes, in order to have all their Vibrations performed in the same Time. For Gravitation (as is shown in the third Book) decreases every way as we go from the Superficies of the Earth upwards in a duplicate Proportion of the several Distances from its Center, but as we go downwards within its Superficies, the Gravitation decreases in a simple Proportion of the several Distances from the Center.

S C H O L I U M.

THE Invention of the Pendulum, is of excellent Use in Astronomy, and where-ever there is occasion to know the true Time; for by this means we have a certain Measure of it, by which we can compute it with the utmost Exactness, by half Seconds or Seconds, or any other parts of Time, according to the different Lengths of the Pendulums; and till this Instrument was found out, there was no certain Measure of Time; for there being no Bodies that we know of, whose Motion is exactly equable, and the parts of Time flowing all equably, whatever moveable Bodies were made
use

use of, they deviated from the true Measure, proportionably to the Inequality of their Motion, so that by this Instrument all others may be regulated, and made fit for common Use. The Pendulum is also of great Use to determine the true Figure of the Body of the Earth; Because the Earth revolves about its own Axis once in twenty four Hours, and because it is not one entire solid Body, but consists partly of a Fluid, and partly of a great Number of different sorts of Bodies which are not closely united to each other; therefore the Figure of the Earth cannot be an exact Globe, but a Sphaeroid, whose Diameter at the æquinoctial Circle is longer than its Axis; for by its revolving about its Axis, all the parts of Matter of which it is composed, except those which lie in the Axis it self, must describe so many concentrick Circles; and because they all describe similar Arches of those Circles in the same Time, therefore their Velocities will be as their nearest Distances from the Axis of the Earth; whence it follows, that some part of the Gravitation of these Bodies of which the Earth is composed, must be taken off by this circular Motion, and that in a certain Proportion to the Swiftness of the Motion; and consequently at the Equinoctial, where the Motion is swiftest of all, the Gravity must be most diminished, the Effect of which must be, that the lighter parts giving way to the heavier, the Earth must be higher at the Equator than at the Poles, where there

is no such Motion. The difference of these Heights, or the difference betwixt the Semidiameter of the Earth at the Poles, and at the Equator, our Author has exactly computed by the help of the foregoing Propositions (*See Prop. XX. Book III. of his Principia.*) and finds it to be about seventeen Miles. For if the Earth be of the fore-mentioned sphæroidal Figure, it is manifest, that as we go from the Equator to the Pole, the Force of Gravitation must increase, because the parts towards the Pole are nearer to the Center than the parts about the Equator; and this difference of Gravity must necessarily affect the Motion of Pendulums, so that two Pendulums of the same Length will not vibrate in the same Time in two different places; and so it is found by Experience, for the length of a Pendulum at *Paris*, exceeds the length of a Pendulum under the Equator, a twelfth part of an Inch, and somewhat above, if they both vibrate in the same Time. By thus comparing the Force of Gravity with the different Lengths of Pendulums, the exact Figure of the Earth may be determined.

SECTION

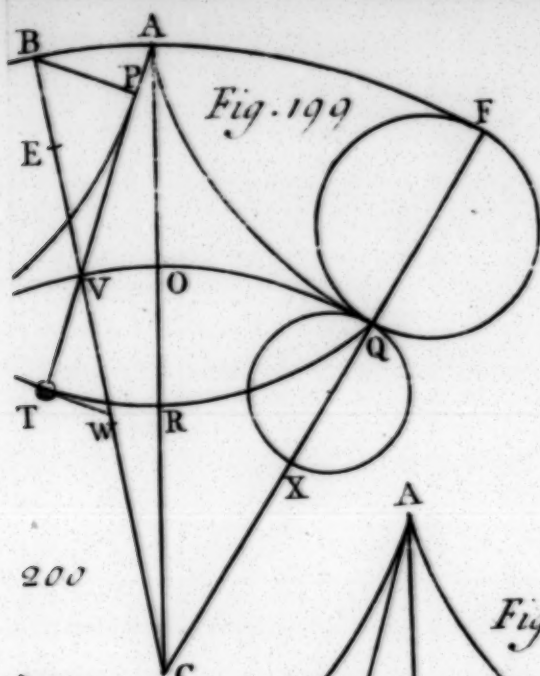


Fig. 199

200

> Q

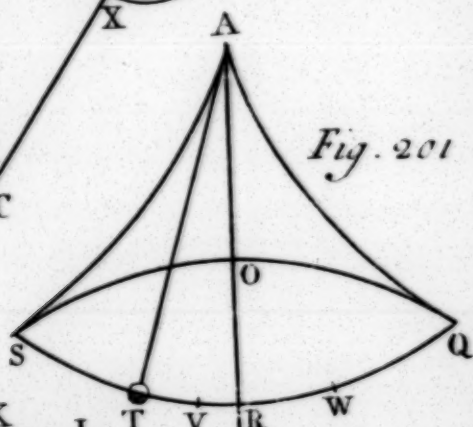


Fig. 201

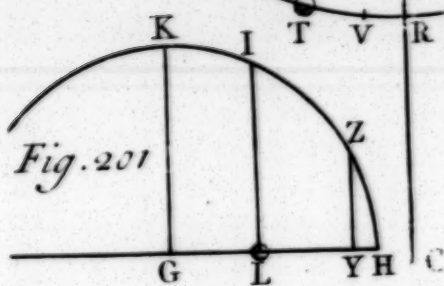


Fig. 201

> q

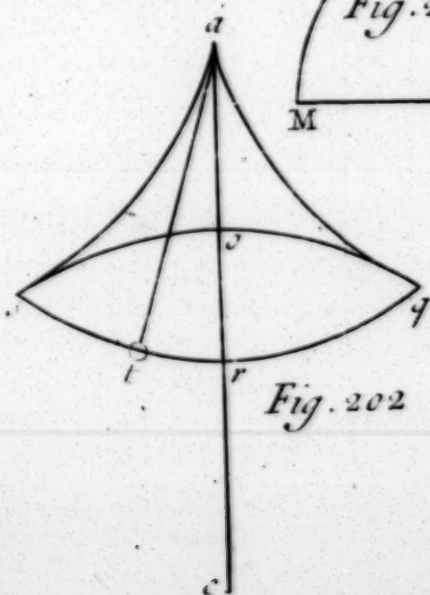
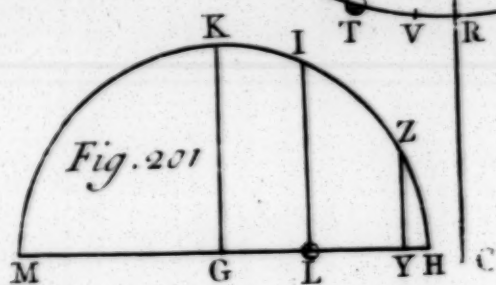
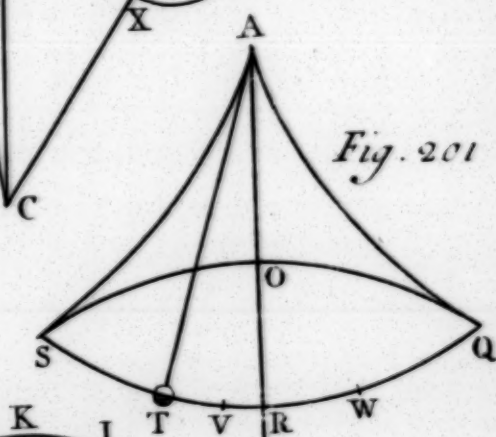
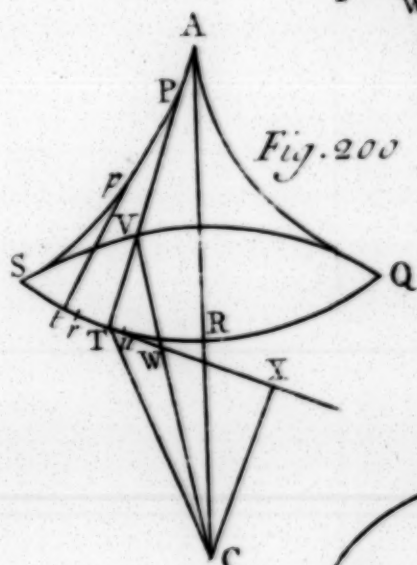
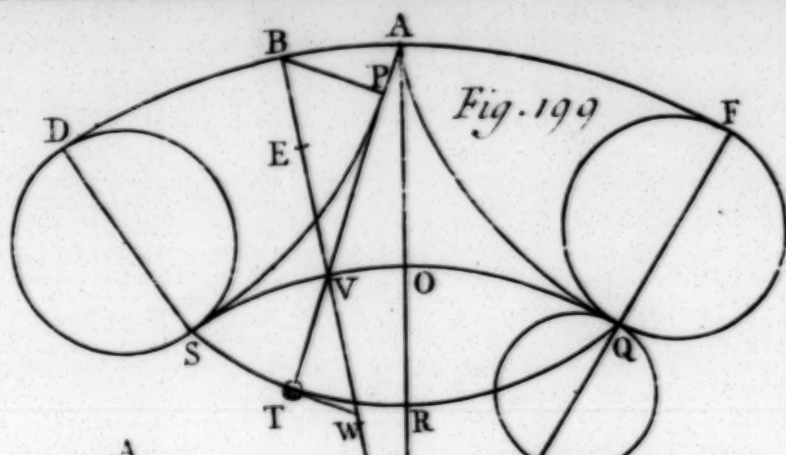
202

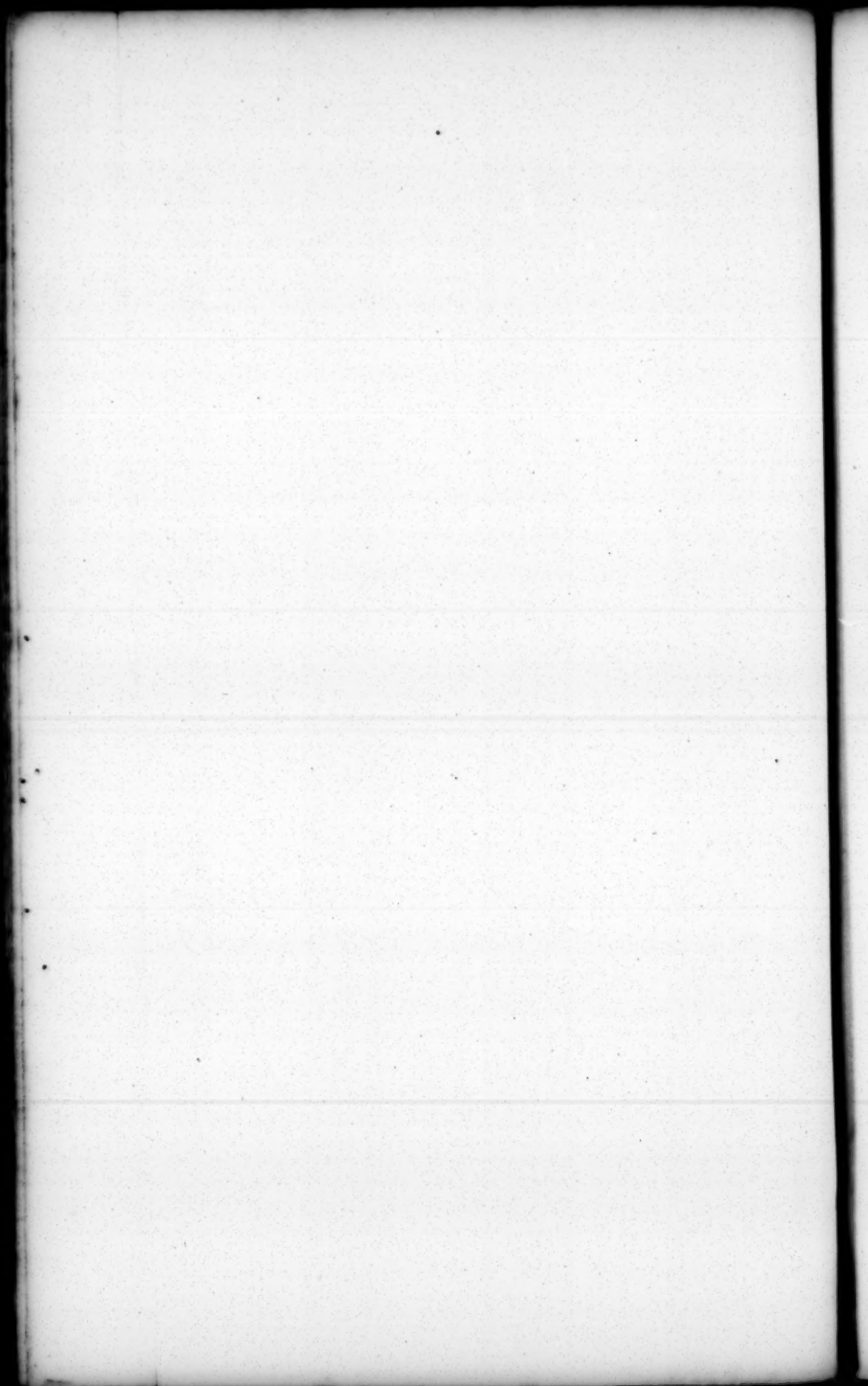


Fig. 202

these
dia-
the
by
See
and
if
idal
the
ion
the
arts
gra-
en-
me
in
Ex-
at
an-
ch,
in
rce
en-
be

N







S E C T I O N X I.

Concerning the Motion of Bodies which tend towards each other with centripetal Forces.

Hitherto we have explained the Motions of Bodies which are attracted to an immoveable Center, of which there are very few, if any, Instances in Nature, that is, where they are exactly so. For Attractions are generally towards Bodies, and the Actions of Bodies attracting and of Bodies attracted, are always mutual and equal, that is, all Bodies attract each other equally, by *Law III*; so that neither the attracting nor the attracted Body can continue at rest, if they are two distinct Bodies, but (by the fourth Corollary of the *Laws of Motion*) may both of them, as it were by a mutual Attraction, be revolved about their common Center of Gravity; and if there be a great many Bodies, which are either attracted by a single Body, and which do also attract that single Body, or if they all mutually attract each other; they ought to be so moved amongst one another, that the common Center

of Gravity, should either continue at Rest, or move on in a streight Line. Wherefore I shall now go on to explain the Motion of Bodies that attract each other mutually, by considering their centripetal Forces as Attractions, though perhaps, *physically* speaking, they may more truly be called Impulses; for it is not probable that there is any natural Vertue or Power in the common Center of Forces, but that the revolving Bodies are some way impelled towards that Center: but we are engaged only in what is mathematical, and therefore setting aside all Disputes in natural Philosophy, I shall make use of common Words, so as to be the easier understood by mathematical Readers.

PROPOSITION LVII. THEOREM XX.

Two Bodies that mutually attract each other, describe similar Figures, both about their common Center of Gravity, and also about each other.

Fig. 203. LET S and P be two such Bodies mutually attracting each other, then if their Distance from each other SP be so divided in C, that CP shall bear the same Proportion to SC, that the Body S bears to the Body P, the point C will be the common Center of Gravity of both the Bodies, about which they may be made to revolve (*by Corol. 2. Law III.*) Now these Bodies continuing always the same, the
Distances

Distances of them from the Center C, which is always proportionable to those Bodies, must always continue the same likewise; that is, suppose the Line SP to move into any new Situation, as TQ, and the Proportion of the Body S to the Body P be as A to α ; then it will be every where throughout the whole Revolution of the Bodies S and P about the common Center of Gravity C.

$A : \alpha :: CP : SC :: CQ : TC$. consequent-
ly

$A : \alpha :: CQ : TC$. (by Equality
of Proportion.)

FOR the same Reason, because the Distance SP is given, and the Proportion betwixt CP, SC, and CQ, TC, is every where the same; the Proportion betwixt any of the Distances of the two Bodies S and P from the common Center of Gravity C, and the whole Distances of the Bodies from each other, will be given also, that is, SC will be to SP, and CP will be to SP in a given Proportion, as will also TC be to TQ, and CQ to TQ. Now all these Distances, viz. SC, TC, QC, PC, are carried about their common Term or Center C with an equal angular Motion, that is, the Angle SCT is always equal to the Angle QCP (by the Elements.) because TC, CQ and SC, CP, lying always in a streight Line, cannot alter their Inclination towards each other: But all streight Lines, which are in a given Proportion

to each other, and are carried about their common Terms or Centers with an equal angular Motion, describe similar Figures about those Centers in Planes, which are either at Rest with these Centers, or are moved with any Motion that is not angular; because (*by the Elements.*) all such Figures having one Angle equal, and the correspondent Sides proportionable, are similar; wherefore the Triangle SCT is always similar to the Triangle QCP. Further, from the point T, let the Line Tp be drawn parallel to SP, and with the Radius TQ, let the Arch Qp be described intersecting the Line TP in p. Also from the point Q, let the Line Qs be drawn parallel to the same Line SP, and on the Center Q, with the Distance QT, let the Arch Ts be described intersecting Qs in s. It is manifest, that if, whilst the Body P moves from P to Q, the Body S moves from S to T; to an Eye that is placed in the Body S, this other Body P must seem to have described the Arch Qp, because the Eye is moved in the same Time from S to T; In like manner to an Eye placed in the Body P, whilst the Body S moves from S to T, it must appear to have moved through the Arch Ts, because the Eye will have been carried from P to Q. Now the Lines sQ, SP and Tp being all parallel to each other (*by the Hypothesis.*) the Angles sQT, QCP, QTp, SCT, are all equal (*by the Elements.*) and the correspondent Sides were before shown to be proportionable

to each other; therefore all the Figures, QT , $\text{QT } p$, and QCP , SCT , are similar, and consequently the Bodies S and P , describe about the common Center of Gravity, and about each other, similar Figures. W. W. D.

PROPOSITION LVIII. THEOREM XXI.

If two Bodies attract each other with any Forces whatsoever, and at the same time revolve about the common Center of Gravity: I say, that if either of those Bodies be made to rest, the other Body may be made with the same Forces, to describe about it as a Center a Figure similar and equal to those Figures which the Bodies when both moved, described about each other; that is, if the Bodies S and P move about the common Center of Gravity C , by means of any centripetal Forces whatsoever; by those very same Forces, the Body P may be made to move about S as its Center, or the Body S may be made to move about P as its Center, and each of them to describe the same Figure which they described about each other, whilst they revolved about the common Center of Gravity, that is, a Figure similar to QCP or SCT , and whose correspondent Sides are equal to TQ and SP .

LET the Bodies S and P revolve about the Fig 204.
common Center of Gravity C , and move from
S to

S to T, and from P to Q. From a given point s , let the Lines sp , sq , be always drawn equal and parallel to the Lines SP, TQ; and the Curve pqv , which the point p describes by revolving about the immoveable point s , will be similar and equal to the Curves which the Bodies S and P describe about each other. For sq being parallel to CQ, and sp parallel to CP (*by the Hypothesis.*) the Angle qsp will be equal to the Angle QCP or SCT (*by the Elements.*) The Radius's SP, sp ; TQ, sq , by which the Curves about the Centers S and s , or about the Center P, are described, being also equal, the Curves themselves must be similar and equal (*by the Elements.*) They must also be similar to the Curves ST and PQV, which the same Bodies describe about the common Center of Gravity C, (*by Theorem XX.*) because the Proportions of the Lines SC, CP, and SP or sp to each other are always the same.

Case 1. THIS common Center of Gravity, must either continue at rest, or move on uniformly in a streight Line (*by the fourth Corol. of the Laws of Motion.*) Let us in the first place suppose it to be at rest, and let there be two Bodies placed in s and p , similar and equal to the Bodies S and P, the immoveable one in s , and the moveable one in p ; then let the streight Lines PR, pr , touch the Curves PQ and pq in P and p ; and let CQ and sq be produced to R and r . Now because the Figures CPRQ,

$cprq$,

$cprq$, are similar (as was proved before.) RQ will be to rq as CP to sp ; (because they are the correspondent Sides of similar Figures.) and therefore the Proportion that RQ bears to rq is given, because the Proportion of CP to sp or SP , is given; that is, the same Proportion that the greater Body S bears to the Sum of the Bodies $S + P$. Wherefore if the Force with which the Body P is attracted towards the Body S , and consequently towards the intermediate Center C , were to the Force with which the Body p is attracted towards s in this same given Proportion of RQ to rq , these attractive Forces would in equal Times always attract the Bodies from the Tangents PR , pr , to the Arches PQ , pq , through the Spaces RQ , rq , proportionable to them; (because if the Time be given, the Spaces run through are as the impressed or attractive Forces.) and consequently the latter Force will cause the Body p to move in the Curve pqv , which is similar to the Curve PQV , in which the former Force caused the Body P to move; and their Revolutions will be compleated in the same Times; (by Composition.) But because these attractive Forces do not bear the same Proportion to each other that CP bears to sp , but are equal to each other, (because the Bodies S and s , P and p , are similar and equal, and the Distances SP and sp are equal, by the Hypothesis.) therefore in equal Times the Bodies must be drawn to equal Distances from the Tangents, that is, if

rt

rt be taken equal to QR , then in the same Time that the Body P is attracted to Q , the Body p will be attracted to t ; wherefore in order to have this latter Body p , attracted through a greater Space rq , it will require a longer Time, and that in a subduplicate Proportion of these Distances, that is, the Time taken up in the Bodies being attracted through the Space rt or RQ , must be to the Time taken up in its being attracted through the Space rq ; as $\sqrt{rt}$ to $\sqrt{rq}$; because (by Lemma X.) the Spaces described by such Bodies in the beginning of their Motions are as the Squares of the Times. Wherefore in order to have the Body p take up more Time before it arrives at q , than the Body P does before it arrives at Q ; it must be made to revolve so much the slower: So that if we suppose the Velocity of the Body p to be to the Velocity of the Body P , in a subduplicate Proportion of the Distance sp to the Distance SP ; that is, putting v for the Velocity of the Body p , and V for the Velocity of the Body P .

$$v : V :: \sqrt{sp} : \sqrt{SP}.$$

in order to have the Arches pq , PQ , (which are to each other as sp to CP , the whole Distances from the Center of Gravity,) described in the Times which are in a subduplicate Proportion of those same Distances: then the Bodies p and P , being always attracted with equal Forces, will describe about the quiescent Centers C and s , the similar Figures PQV , pqv , the latter of which pqv , is similar, and equal
to

to the Figure which the Body P describes about the moveable Body S. W. W. D.

Case 2. LET us now suppose the common Center of Gravity, and the whole Space in which the Bodies are moved about each other, to go on together uniformly in a straight Line: then (*by Corollary 6. of the Laws of Motion.*) all the Motions will be performed in this Space, in the same manner as before; and therefore the Bodies will describe about each other the same Figures as before, and consequently they will be similar, and equal to the Figure *p q v*. W. W. D.

Corol. 1. HENCE it follows, that two Bodies which attract each other with Forces proportionable to their Distance from one another, describe concentrical Ellipsis's, both about their common Center of Gravity, and also about each other. If they move only about a common Center, they will describe concentrick Ellipsis's (*by Prop. X.*) and if they also move about each other, they will describe similar Figures (*by this Prop.*) therefore in both Cases they will do the same. And on the other hand, if such Figures are described, the Forces will be proportionable to the Distances. For if the Time be given, the Forces are to each other as RQ to TO, that is, as CP to TC, because the Figures RQCP, TOSC, are similar.

Corol. 2. AND for the same Reason, two Bodies attracting each other with Forces reciprocally

proccally proportionable to the Square of their Distance from each other, will describe about their common Center of Gravity and about each other, Conick Sections, whose Focus is in the Center, about which the Figures are described. And on the other hand, if such Figures are described, the centripetal Forces are reciprocally proportionable to the Squares of the Distance (*by Prop. XI, XII, XIII.*)

Corol. 3. ANY two Bodies revolving about a common Center of Gravity, will describe Areas proportionable to the Times, by Rays drawn to that Center, and to each other. For the Figures they describe in the same Time being all similar, the Case is the same as if they revolved about the same Center.

PROPOSITION LIX. THEOREM XXII.

If two Bodies S and P revolve about their common Center of Gravity C; the periodical Time of either of them about that Center will be to the periodical Time of either of them, suppose P, revolving about the other S as an immoveable Center, and describing a Figure similar, and equal to the Figures which the Bodies S and P describe about each other, in a subduplicate Proportion, of one of the Bodies S, to the Sum of the Bodies S + P.

FOR by the Demonstration of the foregoing Proposition, the Times in which any
similar

similar Arches PQ and pq are described, are in a subduplicate Proportion of the Distances CP and SP or sp ; that is, in a subduplicate Proportion of the Body S to the Sum of the Bodies $S + P$ (*by the Hypothesis.*) Wherefore, by Composition, the Sums of all the Times in which all the similar Arches PQ and pq are described, that is, the whole Times in which the whole similar Figures are described, must be in the same subduplicate Proportion. $W.W.D.$

PROPOSITION LX. THEOREM XXIII.

If two Bodies S and P attract each other with Forces reciprocally proportionable to the Square of their Distance, and revolve about their common Center of Gravity C : I say, that the principal Axis of the Ellipsis which either of the Bodies, suppose P , describes about the other Body S by this Motion, will be to the principal Axis of the Ellipsis which the same Body P may be made to describe about the other Body S , being at rest, in the same periodical Time; as the Sum of the two Bodies $S + P$, to the first of two mean Proportionals between this Sum and that other Body S .

LET the two Bodies S and P revolve about Fig. 205, the common Center of Gravity C ; and let the point p describe about the immoveable Center s an Ellipsis $pqc p$ similar, and equal to the Ellipsis's

lipfis's which the Bodies S and P describe about each other; then if t be put for the Periodical Time in which the Bodies S and P revolve about the common Center of Gravity C; and T be put for the periodical Time in which the point p revolves in the Ellipsis $pqc p$; it will be, $t : T :: \sqrt{S} : \sqrt{S + P}$. (*by the foregoing Prop.*)

This is upon Supposition that the Ellipsis $pqc p$ is equal to the Ellipsis which the Body P describes about the Body S; but if the periodical Times are to be made equal, then T must be diminished in the Proportion of $\sqrt{S}$ to $\sqrt{S + P}$, that is, in the subduplicate Proportion of S to $S + P$, or, $:: S^{\frac{1}{2}} : (S + P)^{\frac{1}{2}}$; and further, because the principal Axis's of the Ellipsis's are in a sesquialterate Proportion of the periodical Times (*by Prop. XV.*) therefore if the Axis CP be taken in the same Proportion to the Axis cp , that the Sesquialterate of the Subduplicate of the Body's $S + P$, bears to the Sesquialterate of the Subduplicate of the Body S; and on this Axis CP be described the Ellipsis CPS; the periodical Time of the Body P about the other quiescent Body S as its Center, will be equal to the periodical Time in which they revolved about the common Center of Gravity. that is,

$$cp : CP :: \overline{S + P}^{\frac{1}{2}} : S^{\frac{1}{2}}.$$

because the Sesquialterate of the Subduplicate, is equal to the Subtriplicate, or $\frac{2}{3}$ of $\frac{1}{2}$ is equal to

to $\frac{1}{3}$. Now the Cube Root of any two Terms, is the first of two mean Proportionals between those two Terms, and therefore the principal Axis CP of the Ellipsis PCS, will be to the principal Axis of the Ellipsis $pqc p$, as the first of two mean Proportionals between $S + P$ and S, to $S + P$. And on the other hand, cp the principal Axis of the Ellipsis described about the moveable Body, will be to CP, the principal Axis of the Ellipsis described about the immoveable Body, as $S + P$, to the first of two mean Proportionals between $S + P$ and S. W.W.D.

S C H O L I U M.

IF we apply the foregoing Propositions to the present System of the Planets, they will give us a very clear Notion of the Beauty and Simplicity of it. We shall see how regularly and exactly all the primary and secondary Planets perform their Revolutions, in similar Figures, under the Direction of one obvious and uniform Law. Thus (*by Rule III.*) the Matter of which all these Planets are composed, being in general the same, the Effects produced by their mutual Application to each other, must be the same also, or under like Circumstances, the Effects must be similar; thus if we find by Experience, (*See the foregoing Figures.*) that if two Bodies S and P; the first of ten pound Weight, and the other of one pound

T

Weight,

Weight, be connected together by any Thread or Wire of the Length SP ; and this Length be divided into ten equal parts, one of which is equal to SC ; and then these two Bodies be suspended upon the Center C , they will continue in Equilibrio, or if they be put in Motion, they will freely move about the common Center of Gravity C , and about each other, describing similar Figures; if, I say, this holds true in this Case of ten pound Weight, and one pound Weight, it must for the same Reason hold true of twenty pound Weight, and two pound Weight, or of twenty hundred and two hundred, or twenty thousand and two thousand, or any proportionable Sums whatsoever, and also of all Distances reciprocally proportionable to those Sums; that is, whether CP and SC , be taken in Inches, Yards or Miles. If therefore we suppose S to be the Earth, and P to be the Moon, and the Distance SP , so to be divided in C , that CP shall be to SC , as the Quantity of Matter contained in the Earth, is to the Quantity of Matter contained in the Moon; and that they were suspended upon the Arms of a Balance supported at C , it is manifest, they would continue in Equilibrio, or if they were put in Motion according to the Direction of their Tangents in the points P and S , they would revolve about each other in the manner just now described. And if we suppose further, that Instead of being supported at the point C , they

2

were

were in empty Space, out of the Reach of the Attraction of any other Bodies; and instead of being connected with each other by a Balance, they were hindred from flying off in their Tangents, or were kept at their proportionable Distances from each other by the Force of Gravitation: It is manifest, that their Motions would be exactly the same as we have now described. The Center of the Earth therefore is not the Center of the Moon's Motion, but both the Earth and the Moon, move about each other, and about the common Center of Gravity; the Distance of which from the Center of the Moon is to the Distance of the Center of the Earth from the Center of the Moon, as 39,788. to 40,788. The same thing is true of *Jupiter* and his Satellities, and of *Saturn* and his Satellities; and it holds true of all the Planets and the Sun. For since the Quantity of Matter in the Sun, is to the Quantity of Matter in *Jupiter*, as 1067 to 1; and the Distance of *Jupiter* from the Sun, is to the Semidiameter of the Sun, in a Proportion very little greater than this; the common Center of Gravity of *Jupiter* and the Sun, will be in a Point a little above the Superficies of the Sun. By the same Argument, because the Quantity of Matter in the Sun, is to the Quantity of Matter in *Saturn*, as 3021 to 1; and the Distance of *Saturn* from the Sun, is to the Semidiameter of the Sun, in a Proportion which is a little less than this; therefore the common Center of Gravity of *Saturn* and the Sun, is in a Point a little within the

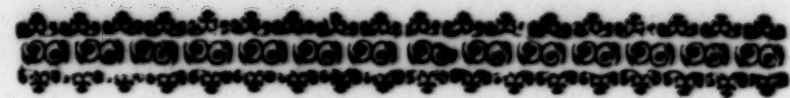
Superficies of the Sun. By the same Method of Computation, if the Quantity of Matter contained in all the Planets and their Satellities together, supposing them to be all of the same side of the Sun, be compared with the Quantity of Matter contained in the Sun; the common Center of Gravity of the whole System would be scarce above one whole Diameter of the Sun, distant from the Sun's Center. In other Cases, where some of the Planets are on one side of the Sun, and some on another; the Distances of the common Center of Gravity from the Sun's Center, must be always less than the Diameter of the Sun. Whence it follows, that because the Center of Gravity of the whole System, must always continue at Rest, and the Sun and Planets move always round about it, and about each other, at different Distances, according to the various Positions they are in with respect to each other; therefore the Sun is never very far distant from this Center. And this common Center of Gravity of the Earth and Sun, and all the Planets, ought to be looked upon as the Center of the World. For since the Earth, the Sun, and all the Planets, gravitate towards each other, and therefore, in Proportion to the Force of their Gravity, are continually altering their places, or are in perpetual Agitation, according to the Laws of Motion; it is manifest, that *their* Centers, which are always in Motion, cannot be accounted the Center of the World, which

is

is always at Rest. If indeed that Body, to which all other Bodies gravitate most, is to be placed in the Center of the World, (as the common Opinion is,) then this Privilege must be given to the Sun. But since it is certain, that the Sun it self moves, we must find out a Point which does not move, and from which the Center of the Sun moves as little as can be, and from which it would move still less, if the Sun were denser and bigger, so as to be harder to move at all.

By these Propositions we are also led to find out the Proportion that there is between centripetal Forces, and the central Bodies to which those Forces are usually directed. For it is very reasonable to think, that the Forces which are directed towards any Bodies, depend upon the Nature and Quantity of those Bodies, as we see by Experience in Loadstones, whose different attractive Forces depend upon the Bigness and Constitution of such Stones. And in all such Cases as these, the Attractions of the Bodies, are to be estimated, by ascribing the proper Forces to every single Particle of which they are composed, and then collecting the Sum of those Forces. The Word *Attraction*, is here to be understood in a general Sense, for any *Endeavour* or *Tendency* that there is in Bodies to approach towards each other, whether that *Endeavour* proceeds from the Action of the Bodies either tending towards each other, or agitating each other by some

Effluvia which they send forth; or whether it be caused by the Action of some Æther, or Air, or any kind of Medium, corporeal or incorporeal, which some way or other impels the Bodies swimming in it, towards each other. The Word *Impulse*, is to be understood in the same general Sense; for I do not in this Treatise consider the *physical* or *natural* Sorts or Qualities of Forces, but only their Quantities and mathematical Proportions, as was explained in the Definitions at the beginning of the Book. It is the Business of Mathematicks, to find out the Quantities of the Forces, and those Proportions which flow from any Conditions whatsoever that are laid down or supposed: And after this, when we come to Physicks or Natural Philosophy, then these Proportions are to be compared with the Phænomena; that we may see what kind of Forces will agree with the several Kinds of attractive Bodies. When we have gone thus far, we may with greater Safety dispute about the several sorts of Forces and their natural Causes and Proportions. Let us therefore now examine with what sort of Forces, spherical Bodies, composed of such sort of attractive Particles as we have been describing, ought to act upon each other, and what sort of Motions will follow from thence.



THE
 TWELFTH SECTION
 OF THE
 FIRST BOOK
 OF

Sir *ISAAC NEWTON*'s

Principles Explain'd.

Of the attractive FORCES of SPHÆRICAL
 BODIES.

THE Design of the following Section,
 is to demonstrate the Laws of Attraction in sphærical Bodies with respect to each other's Centers, or to compute the Forces by which they tend towards each other's Centers, measured by the Distances from those Centers. And in order to this, the sphærical Bodies are supposed to be composed of an infi-

nite Number of physical Particles, or very small Corpuscles equal to each other; and that every one of these Corpuscles is endued with the same Degree of attractive Force; or that equal centripetal Forces tend towards each of them; and that these Forces increase or decrease in some certain Proportion to the Distances from the Corpuscles; hence it is required to determine what the Result of such a Combination of Forces will be, in two or more spherical Bodies of equal or unequal Magnitudes, of the same or different Densities, with regard to the Centers of those spherical Bodies. But because these centripetal or attractive Forces, may increase or decrease in all possible Variety of Suppositions, that is, they may increase in Proportion to the Distances, directly, or to the Squares, Cubes, or any other Multiples of the Distances from the Corpuscles to which they tend, or they may decrease as the Distances, the Squares, Cubes, &c. of the Distances: Therefore we shall consider only two of these Cases. First, where the centripetal Forces decrease as the Squares, or in a duplicate Proportion of the Distances from the Corpuscles. And Secondly, where the centripetal Forces increase in Proportion to their Distances from the Corpuscles to which they tend; because these are the two most common Cases that happen in Nature.

PROP-

PROPOSITION LXX. THEOREM XXX.

If to every Point of a sphærical Superficies, there tend equal centripetal Forces, decreasing as the Squares of their Distances from those Points: I say, that a Corpuscle placed any where within such a Superficies, will not be attracted any way by these Forces, but will continue at Rest.

HERE we begin with the most simple Case, and conceive a sphærical Body to be composed of an infinite Number of sphærical concentrick Superficies's, or very thin Shells, whose Thickness is so small, as not to enter into the Computation; and we first consider one of these sphærical Superficies or Shells, and suppose every Point of this Superficies to attract a Corpuscle placed any where within it, but not the Corpuscle to attract the several Points of the Superficies.

LET HIKL represent this sphærical Superficies, and P the Corpuscle placed within it. Fig. 207. Through the point P, to this Superficies, let the two streight Lines HK, LI be drawn, including the two very small Arches HI, KL; then, because the Triangles HPI, LPK, are similar (*by the Elements, and by Lem. III. Cor. 3.*) these Arches will be proportionable to the Distances HP, LP; and the Particles of the sphærical Superficies, of which these Arches are

are the Diameters, and which are every way terminated by streight Lines passing through the point P, and which may be conceived to constitute very small similar Cones, whose Vertices terminate at P, and their Bases at the Surfaces: These Particles or Bases being Circles, are (*by the Elements.*) as the Squares of their Diameters HI, KL or HP, PL. Therefore the Forces of these Particles upon the Corpuscle P, are equal, for they are (*by the Hypothesis.*) as the Particles directly, and the Squares of the Distances inversely, and these two Proportions are equal: The Attractions therefore exerting themselves equally contrary ways, destroy each other. By the same Argument, all the Attractions over the whole Surfaces are destroyed by contrary Attractions; and therefore the Particle P cannot be impelled any way by these Attractions. Thus let C, c, be put for the two small sphaerical Surfaces at KL and HI, let F, f be put for their Forces, and let D, d be put for the Distances from P. Then $C : c :: HL^2 : HI^2 :: PL^2 : HP^2$. But (*by the Hypothesis.*) $F : f :: \frac{C}{PL^2} : \frac{c}{HP^2}$. then putting PL^2, HP^2 for C, c; It will be $F : f :: \frac{PL^2}{PL^2} : \frac{HP^2}{HP^2}$. Therefore $F = f$.

Fig. 207. *Corol. 1.* IF the Corpuscle P, attracts each of the small Surfaces's KL, IH, in the same manner as they attract it; the whole Attraction will,

will be double what it was before, but the same Proportion of Equality will continue.

Cor. 2. THIS being the Case of our Earth, if we suppose it hollow, or a Shell or Crust of a given Thickness, any Body placed any where within it would of it self remain at Rest, where-ever it was placed, would have no Weight, might be moved with the least Force, would continue in Motion with a given Velocity and in a given Direction till it was stopped by the inward Superficies of the Earth's Crust.

PROPOSITION LXXI. THEOREM XXXI.

The same things being supposed as in the foregoing Proposition; I say, that a Corpuscle placed any where without the sphaerical Superficies, is attracted by a Force which is reciprocally proportionable to the Square of its Distance from the Center of the sphaerical Superficies; that is, the Force decreases as the Square of the Distance from that Distance.

HERE AHKB, *ahkb*, are two equal Superficies, or rather the same Superficies represented twice, and these Superficies are supposed to be generated by the equal Arches HI, *bi*, and KL, *kl*, turning round upon the Diameters AB, *ab*; then the small physical Particles HI, *bi*, multiplied by the Circles thus generated, or (because Circles are as the *Radii*) multiplied by

Fig. 202.

by the Radii QI, qi ; will represent the Quantity of Matter contained in one of the physical Rings, of which these sphærical Superficies are composed; and the Force of Attraction, is (by the Hypothesis.) as this Quantity of Matter directly, and the Square of its Distance from the Particle attracted inversely; that is, as $\frac{HI \times IQ}{HP^2}$, or $\frac{bi \times iq}{bp^2}$. Let us see now what

Fig. 208. the same Force is with respect to the Center of the whole Superficies S or s . Let $AHKB, abkb$, be two equal sphærical Superficies described on the Centers S, s , and the Diameters AB, ab ; and let P, p , be two Corpuscles placed without these Superficies in their Diameters produced. Let the Lines PHK, PIL, pbk, pil , be drawn from those Corpuscles, cutting off the equal Arches HK, bk , and IL, il , from the great Circles AHB, abb ; And let SD, fd ; SE, fe ; IR, ir , be drawn perpendicular to those Lines, so that SD, fd , may cut PL, pl , in F and f : Also let IQ, iq , be drawn perpendicular to the Diameters. Let the Angles DPE, dpe , vanish or be infinitely small; then because DS is equal to ds , and ES equal to es , their Differences, viz. the very small Lines DF, df , are also equal; and the Lines PE, PF , and pe, pf , may be looked upon as equal, because their last Ratio or Proportion when the Angles DPE, dpe , vanish together, or when their Sides coincide, is a Ratio of Equality. These things

things being thus; Because the Triangles $PR I$, PDF , are similar, and the Triangles pri , pdf , are similar also; Therefore

$$PI : PF :: RI : DF.$$

$$\text{and } pf : pi :: df \text{ or } DF : ri.$$

Therefore (by

$$\text{Composition.}) PI \times pf : PF \times pi :: RI : ri.$$

$$:: HI : bi.$$

that is, as the Arch $I H$ is to the Arch $i b$, (by *Corol. 3. Lem. VII.*) Again, because the Triangles PSE , PIQ , are similar, and pse , $p iq$, are also similar, therefore

$$PI : PS :: IQ : SE.$$

$$ps : pi :: se \text{ or } SE : iq.$$

Therefore (by

$$\text{Composition.}) PI \times ps : PS \times pi :: IQ : iq.$$

Then multiply all these Ratio's together, and it will be as $PI^2 \times pf \times ps$, to $pi^2 \times PF \times PS$; so is $HI \times IQ$, to $bi \times iq$. That is, the circular Superficies which the Arch $I H$, describes by the Revolution of the Semicircle $A K B$ about the Diameter $A B$; is to the circular Superficies which the Arch $i b$, describes by the Revolution of the Semicircle $a k b$ about the Diameter $a b$; as $PI^2 \times pf \times ps$, to $pi^2 \times PF \times PS$. And the Forces with which these Superficies attract the Corpuscles P, p , in the Direction of the Lines PI, pi , are (by the *Hypothesis*.) as the Superficies directly, and the Squares of the Distances of the Corpuscles from the Superficies inversely; that is, putting F, f , for these Forces.

$$F : f$$

Of the attracting FORCES

$$F : f :: \frac{PI^2 \times pf \times ps}{PI^2} : \frac{pi^2 \times PF \times PS}{pi^2}.$$

that is, $F : f :: pf \times ps : PF \times PS$.

This is the Proportion of the Forces as they tend towards the circular Superficies's in the Directions PI, pi : But (by the resolving of Forces according to *Cor. 2. of the Laws of Motion.*) these Forces in the Directions PI, pi , are to the Forces in the Directions by which they tend to the Centers PS, ps , as PI to PQ , and pi to pq , that is, (because of the similar Triangles PIQ and PSF ; piq and psf ;) as PS to PF , and ps to pf . Therefore in the foregoing Proportion of the Forces, $pf \times ps$ to $PF \times PS$, if these Terms be multiply'd by $\frac{PF}{PS}$, and $\frac{pf}{ps}$; they will become $\frac{PF \times pf \times ps}{PS}$

to $\frac{pf \times PF \times PS}{ps}$; that is, the Attraction of the Corpuscle P towards S , will be to the Attraction of the Corpuscle p towards s ; as $\frac{PF \times pf \times ps}{PS}$, to $\frac{pf \times PF \times PS}{ps}$. or as $PF \times pf$

Fig. 208. $\times ps$ square to $pf \times PF \times PS$ square, that is, (dividing by $PF \times pf$;) as ps square, to PS square. And by the same Argument, the Forces by which the Superficies's generated by the Revolution of the Arches KL, kl , attract the Corpuscles P, p ; will be as ps square, to PS square; and the same Proportion will hold for the Forces of all the circular Superficies's

ficies's that each of these sphærical Superficies's can be resolved into, by taking sd always equal to SD , and se equal to SE . And (by Composition.) the Forces of the whole sphærical Superficies's, will be in the same Proportion. W. W. D.

PROPOSITION LXXII. THEOREM XXXII.

If to every Point of any Sphære there tend equal centripetal Forces, decreasing as the Squares of the Distances from those Points, and the Density of the Sphære be given, and the Proportion of the Diameter of the Sphære to the Distance of a Corpuscle from its Center, be also given: I say, that the Force with which the Corpuscle is attracted, is proportionable to the Semidiameter of the Sphære.

LEMMA to this PROPOSITION.

THE Sphæres AHB , ahb , may be resolved into similar Particles, placed in a similar Position, Fig. 229. in the following manner. Let PS , ps , the Distances of the Particles P , p , from the Centers of the Sphæres S , s , be taken proportionable to the Diameters AB , ab ; then suppose these Diameters divided into an infinite Number of equal parts KL , kl ; on the Centers P , p , and with the Distances PK , PL , and pk , pl , describe the concentrick Arches KI , LH , and ki , lh .
Then

Then PS, pf , the Distances of the Particles from the Centers of the Sphæres, are proportionable to the Diameters (*by the Hypothesis.*) and the Distances of the points K, L , and k, l , from the Centers S, s , are also proportionable to the same Diameters; therefore (*by Composition.*) the Distances of the points K, L , and k, l , from the Corpuscles P, p , are proportionable to the Diameters; and so likewise are the Distances of the Particles I, H , and i, h , from the same Corpuscles P, p . And the same holds true for all other points in the Diameters, and (*by Composition*) for the whole Sphæres. W. W. D.

Fig. 209. SUPPOSE two Corpuscles, each of them to be attracted by a different Sphære, one Corpuscle by the one Sphære, and the other Corpuscle by the other Sphære, and that their Distances from the Centers of their respective Sphæres, are proportionable to the Diameters of these Spheres, and also that the Sphæres are resolved or divided into similar Particles, placed in a similar manner, or in a like Position with respect to each Corpuscle. Then the Attractions of the one Corpuscle towards the several Particles of the one Sphære, will be to the Attractions of the other Corpuscle towards the same Number of proportionable Particles of the other Sphære, in a Proportion compounded of the Proportion of the Particles directly, and the Square of their Distances inversely (*by the Hypothesis and by Composition.*) But the Particles

I

are



cles
or-
(is.)
k, l,
ble
po-
and
ble
the
om
olds
ers,
res.

em
or-
or-
eir
ive
ers
are
ced
re-
of
les
ons
m-
ner
the
the
y-
les
are

Fig. 203

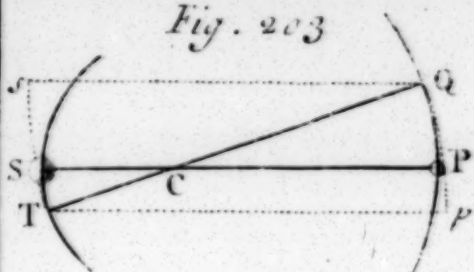


Fig. 204

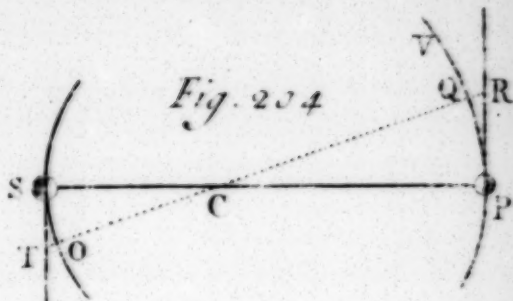


Fig. 205

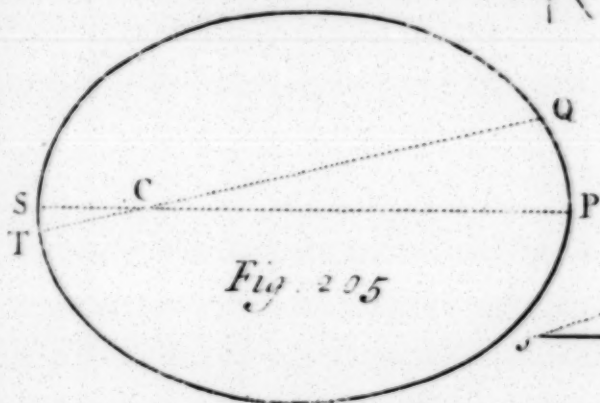


Fig. 204

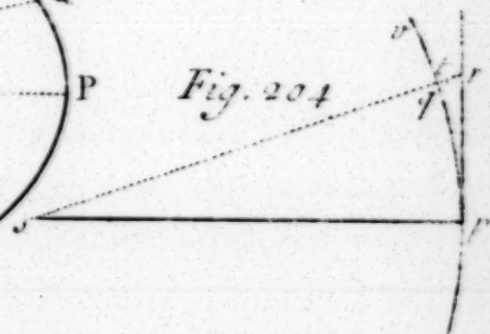


Fig. 205

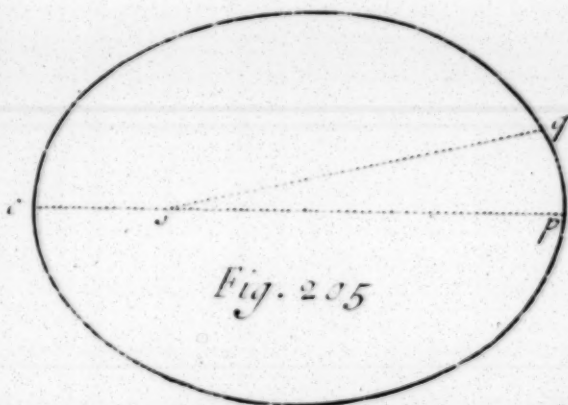


Fig. 206

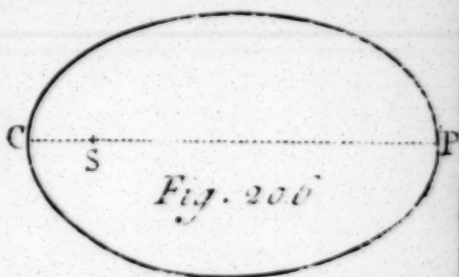


Fig. 208

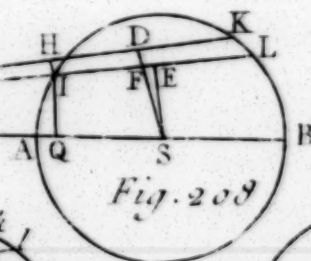


Fig. 208

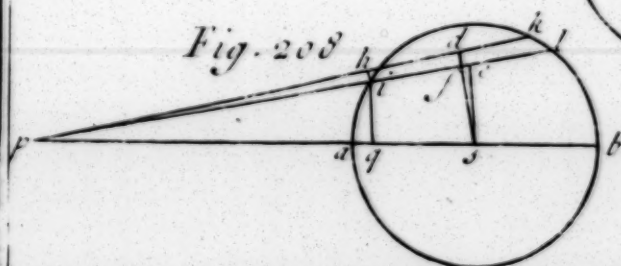
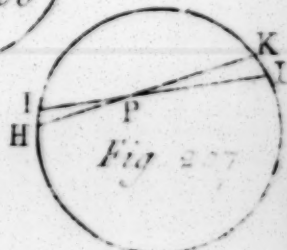


Fig. 207



a
t
h
/

are as the Sphæres, that is, as the Cubes of their Diameters, (*by the Elements.*) and the Distances are as their Diameters simply (*by the foregoing Lemma.*) therefore, the direct Proportion of the Cubes of the Diameters, divided by the Squares of the Diameters, gives the Proportion of Diameter to Diameter.

LET the two Sphæres be AHB, *abb*, and the Corpuscles P, *p*; let the Diameter of the Sphære AHB, be R, and the Diameter of the Sphære *abb*, be *r*; the Distances of the Corpuscles from their respective Sphæres D, *d*, and the Forces by which they are attracted F, *f*. Then because the Attraction of the Corpuscle P, is to the Attraction of the Corpuscle *p*, in a Proportion compounded of all the Particles of which the Sphære AHB is made up of, directly, that is, as the Cube of its Diameter, *viz.* R^3 , and the Squares of their Distances reciprocally, or as the Square of the Distance of P from the Center of the Sphære AHB reciprocally, *viz.* $\frac{1}{D^2}$: And so for the Particle *p*.

Therefore $F : f :: \frac{R^3}{D^2} : \frac{r^3}{d^2}$. but (*by the Hypo-*

thesis.) $R : r :: D : d$. Wherefore $F : f :: \frac{D^3}{D^2}$

$: \frac{d^3}{d^2}$, that is, (*by Division.*) $F : f :: D : d$.

W. W. D.

U

Corol. 1.

Fig. 209. *Corol. 1.* HENCE it follows, that if the Corpuscles P and p , were made to revolve about the Sphæres AHB, abb , made up of equally attractive Matter, and the Distances of these Particles be proportionable to the Diameters of the Sphæres; their periodical Times would be equal.

Corol. 2. AND on the contrary, if their periodical Times be equal, their Distances will be proportionable to their Diameters. These two Corollaries follow from *Prop. IV. Cor. 3.*

Corol. 3. IF to every Point of any two similar Solids, which are equally dense, there tend equal centripetal Forces decreasing as the Squares of the Distances from those Points; the Forces by which Corpuscles placed in a similar Position with respect to those Solids, will be attracted by them, will be as the Diameters of each of these Solids. For the Solids are as the Cubes of their Diameters (*by the Elements.*)

PROP. LXXIII. THEOREM XXXIII.

If to every Point of any given Sphære, there tend equal centripetal Forces decreasing in a duplicate Proportion, or as the Squares of the Distances from those Points: I say, that a Corpuscle situate any where within the Sphære, will be attracted with a Force proportionable to its Distance from the Center.

LET

LET the Corpuscle P, be placed within the Sphære ACBD, described on the Center S; Fig. 210. and on the same Center S, with the Distance SP, let the Sphære PEQF be described within the former Sphære. It is evident (*by Prop. LXX.*) that the several concentrick Superficies, of which the Difference of the Sphæres, *viz.* AEBF is composed, will have no Effect at all upon the Particle P; because the Attractions of all the Points of which they are composed, are destroyed or ballanced by contrary Attractions equal to them. So that there remains only the Attraction of the internal Sphære PEQF on the Corpuscle P. And this is proportionable to the Distance from S (*by Prop. LXXII.*) W. W. D.

S C H O L I U M.

THE Superficies of which Solids are composed, are not considered in these Propositions, as merely mathematical; but as Orbs so very thin, that their Thickness is as it were nothing, that is, as vanishing Orbs of which the Sphære is at last composed, when the Number of the Orbs is increased, and their Thickness's diminished infinitely: In the same manner, by Points of which Lines and Superficies and Solids are said to be composed, we are understand, equal Particles, whose Magnitude we have no Regard to.

PROP. LXXIV. THEOREM XXXIV.

The same things being supposed as before; I say, that a Corpuscle placed any where without the Sphære, is attracted by a Force reciprocally proportionable to the Square of its Distance from the Center of the Sphære.

FOR let the Sphære be distinguished into an infinite Number of concentrical sphærical Superficies's, the Attraction of every one of these Superficies's upon the Corpuscle, will be reciprocally proportionable to the Distance of the Corpuscle from the Center of the Sphære, which is the common Center of all the Superficies's (*by Prop. LXXI.*) Therefore (*by Composition.*) the Sum of all the Attractions of the whole Sphære upon the Corpuscle, will be in the same Proportion. W. W. D.

Corol. I. HENCE it follows, that if the Sphæres be homogeneous, or of the same kind, in equal Distances the Attractions are as the Sphæres, that is, as the Quantities of Matter. For (*by Prop. LXXII. see the Figure there.*) if the Distances are proportionable to the Diameters of the Sphæres, the Forces will be as the Diameters. Let the greater Distance be diminished in this Proportion, that the Distances may become equal, and the Attraction will be increased as the Square of that Proportion, that is, it will be in a triplicate Proportion,

tion, or as the Cube of the other Attraction, that is, it will be in Proportion to the Sphæres.

Thus (*using the same Sybols as in Prop. 72.*)

IF the Distances be unequal $F:f::R:r$. Let PS be diminished in this Proportion, that is, $PS:ps::r:R$. then will PS be equal to ps , but its Attraction will be increased as PS square is diminished, that is, $F:f::RD^2:rd^2$. that is, because $R:r::D:d$. It will be, $F:f::R^3:r^3$.

Corol. 2. IN all Distances whatsoever, the Attractions will be as the Sphæres directly, and the Squares of the Distances inversely. For if the Distances be given, the Attractions are as the Sphæres $R^3:r^3$, and if the Sphæres be given, the Attractions are reciprocally as the Squares of the Distances $\frac{1}{D^2}:\frac{1}{d^2}$. and if neither of them be given, the Attractions are in a Proportion compounded of both these, that is, $F:f::\frac{R^3}{D^2}:\frac{r^3}{d^2}$.

Corol. 3. IF a Corpuscle placed any where out of an homogeneous Sphære, be attracted by a Force reciprocally proportionable to the Square of its Distance from the Center of the Sphære, and the Sphære consists of attractive Particles, the Force of every one of its Particles will decrease as the Square of their Distances from the Corpuscle. This is only the Reverse of the former.

PROP. LXXV. THEOREM XXXV.

If to every Point of a given Sphære, there tend equal centripetal Forces decreasing as the Square of the Distances from those Points; I say, that any other similar Sphære, is attracted by this Sphære, with a Force reciprocally proportionable to the Square of the Distance of their Centers from each other.

Fig. 212. LET A be the attracting Sphære, and B the attracted Sphære; Then the Attraction of every Particle of the Sphære B, is reciprocally as the Square of the Distance of that Particle from the Center of the attracting Sphære A, (*by Prop. LXXIV.*) and is also the same as if the whole attractive Force of the Sphære A came from one single Corpuscle placed in its Center. Now this Attraction is just as much as the Attraction of that same Corpuscle would be, if on the other hand it were attracted by every Particle of the attracted Sphære B, with the same Force that it attracts them. But that Attraction of the Corpuscle, would be reciprocally proportionable to the Square of its Distance from the Center of the Sphære B. (*by Prop. LXXIV.*) Therefore the Attraction of the Sphære A, which is equal to it, is in the same Proportion, *viz.* as the Square of the Distance A B. W. W. D.

Corol. I. THE Forces with which attracting Sphæres, draw attracted Sphæres, are as the
the

the attracting Sphæres applied to the Squares of the Distances of their Centers from the Centers of the Sphæres which they attract, or they are as the Sphæres directly, and the Squares of the Distances of their Centers from the Centers of the Sphæres which they attract, reciprocally. For, if the Distances be the same, the Forces will be as the Sphæres directly, and if the Sphæres be the same, the Forces will be reciprocally as the Squares of the Distances: therefore (*by Composition.*) the Forces are as the Sphæres directly, and the Squares of the Distances inversely; that is,

$$F : f :: \frac{R^3}{D^2} : \frac{r^3}{d^2}.$$

Corol. 2. THE same thing holds, if the attracted Sphære B attracts the Sphære A also. For then every Point of the Sphære B, attracts every Point of the Sphære A, with the same Force that every Point of the Sphære A, attracts every Point of the Sphære B; and therefore, because in all Attractions, the Point that is attracted and the Point that attracts, are equally forced towards each other (*by Law 3.*) therefore the Force of the mutual Attraction of the Points, is double to what it is when considered in each of them separately, but the Proportion remains the same as before.

Corol. 3. ALL those things which were demonstrated before, concerning the Motion of Bodies about the Focus's of Conick Sections, hold true in this Case also, if the attracting

U 4

Sphære

Sphære be placed in the Focus, and the Bodies be moved any where out of the Sphære.

Corol. 4. AND all those things which were demonstrated concerning the Motion of Bodies about the Centers of Conick Sections, hold good here also, if the Motion be performed any where within the Sphære.

PROP. LXXVI. THEOREM XXXVI.

If there be Sphæres that are any way dissimilar (with respect to the Density of their Matter, and their Attractive Force;) as they go from the Center to the Circumference; but as they go in the Circumference, are every where similar at every given Distance from the Center; and the attractive Force of every Point decreases in a duplicate Proportion of the Distance from the attracted Body: I say, that the whole Force with which one of these sort of Sphæres attracts another, is reciprocally proportionable to the Square of the Distance of their Centers.

Fig. 213. IN order to this, let us suppose a Number of unequal Sphæres, but all of them similar, that is, composed of equal Particles, which Particles are at an equal Distance from each other, and that Distance such, that these Sphæres may be put one within another, and by that means compose one Sphære, of different Densities at different Distances, and of the same

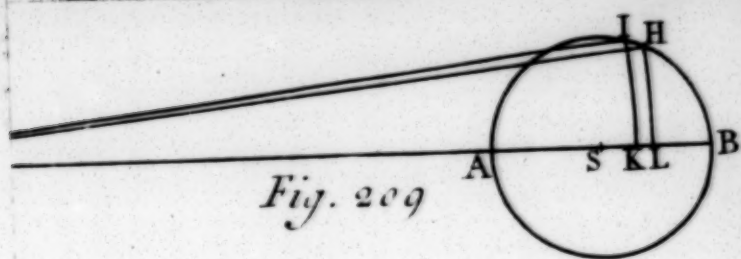


Fig. 209

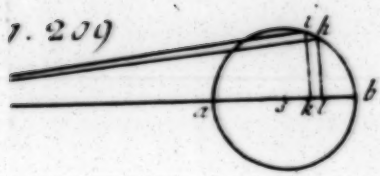


Fig. 209

Fig. 210

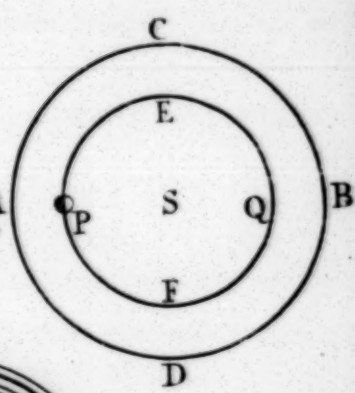


Fig. 211

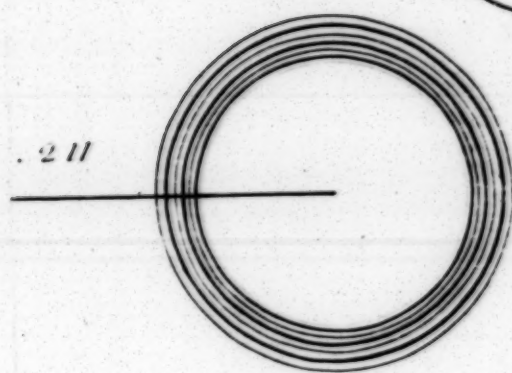
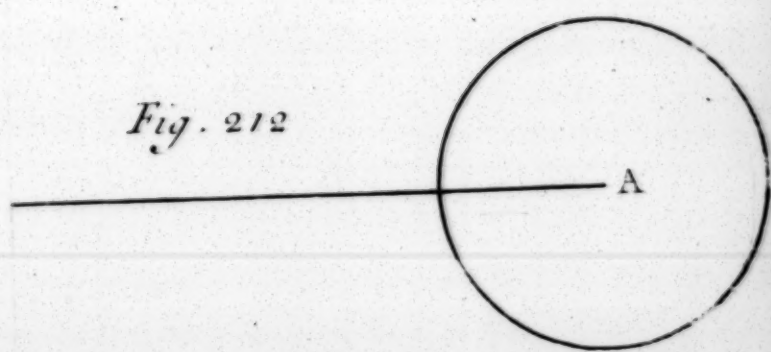


Fig. 212



dies

vere

dies

old

any

I.

ilar

ter,

rom

they

mi-

ter;

de-

Di-

bat

fort

ally

nce

ber

lar,

ich

ach

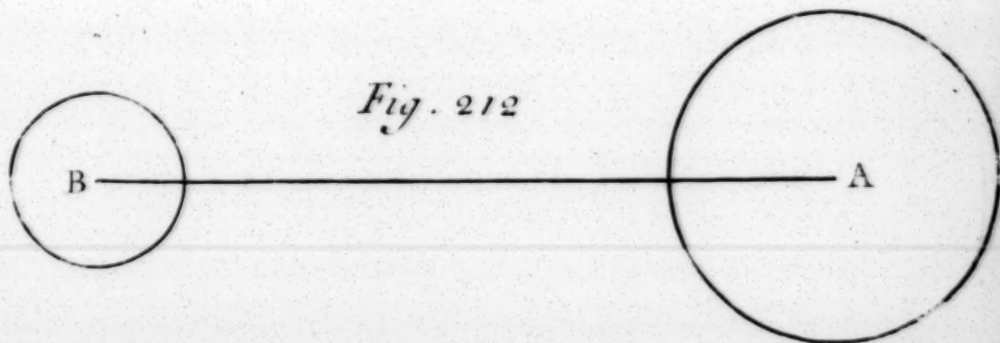
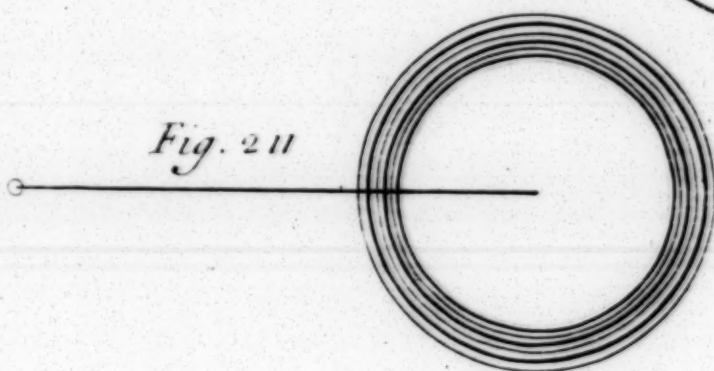
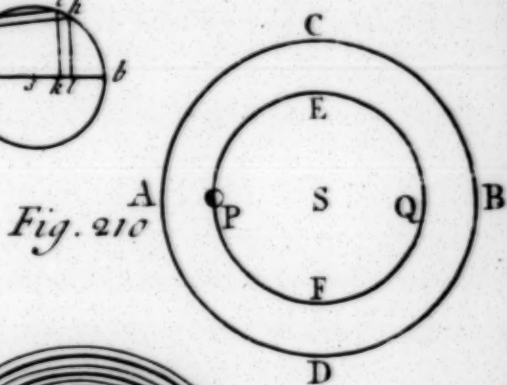
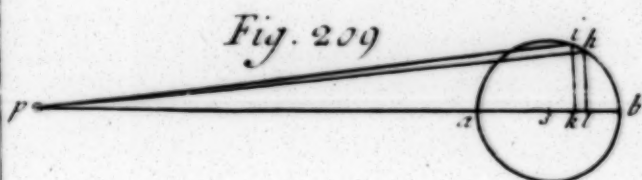
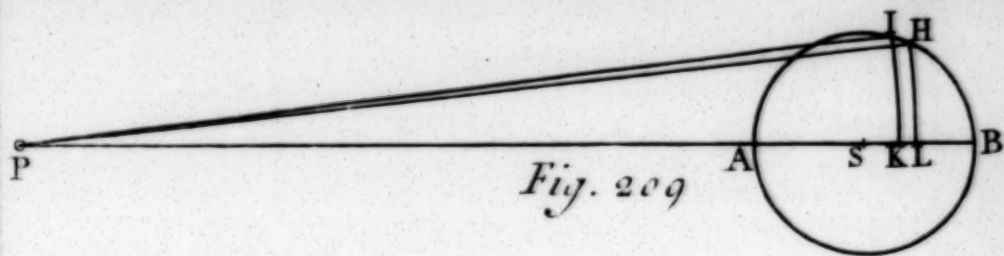
nefe

and

ent

the

me



fa
C
b
a
e
v
e
J
c
t
c
M
S

same Density at any given Distance from the Center. Let $AB, CD, EF, \&c.$ be that Number of Sphæres placed concentrically one within another, so that the internal ones added to the external ones, may make the Matter denser towards the Center, or substracted from those external ones, may make it rarer; then (*by Prop. LXXV.*) these will attract as many other concentrick similar Sphæres GH, IK, LM , that is, each of these singly, will attract every one of the other, with Forces reciprocally proportionable to the Square of the Distance SP ; and (*by Composition or Division.*) the Sum of all these Forces, or the Excess of some of them above the other; that is, the Force with which the whole Sphære AB , composed of any Number of concentrick Sphæres, or of the Difference of concentrick Sphæres, attracts the whole Sphære GH , composed likewise of concentrick Sphæres, or of their Differences, will be in the same Proportion. Let the Number of the concentrick Sphæres be increased infinitely; so that the Density of the Matter, and consequently the attractive Force, may increase or decrease, as it goes from the Circumference to the Center, according to any Law or Proportion whatsoever; and let those places which are not so dense as the rest, be filled up with Matter which has no attractive Power, so that the Sphæres may become of any Form that you please; and the Force with which the one of these Sphæres $AB, CD, \&c.$ thus compound-

compounded, attracts the other GH, IK, &c. will still be by the foregoing Argument, in the same inverse Proportion of the Square of the Distance SP. W. W. D.

Corol. 1. HENCE it follows, that if a great many of the same sort of Sphæres, similar to one another in all respects, do mutually attract each other; the accelerative Attractions of every single Sphære, upon every other single Sphære, or the Celerities which every single Sphære, produces in a given Time, in every other single Sphære, by attracting it; are as the attracting Sphæres, if the Distances of the Centers be the same. Thus, suppose the Sphæres A, B, and C, D, to attract each other mutually, and the Distances from the Centers BA, CD, to be equal. Let the Quantities of Matter (which are as the Cubes of their Diameters) be as 27 in A, 7 in B, 8 in C, and 6 in D; then will the Celerity with which B approaches A in a given Time, be to the Celerity with which D approaches C in the same given Time as 27 to 8.

Corol. 2. AT any unequal Distances, the accelerative Attractions will be as the attracting Sphæres, applied to the Squares of the Distances between the Centers; or they will be as the Sphæres directly, and the Squares of the Distances of the Centers inversely. Thus, let the Distances between the Centers BA and CD be as 4 to 1; then the Celerity with which B approaches

proaches A, is to the Celerity with which D approaches C, as $\frac{27}{16}$ to $\frac{8}{1}$, or as 1,68. to 8.

Corol. 3. AND the whole Motions produced by the Attractions, or the Weights of the Fig. 214. Sphæres upon the Sphæres, are as the attracting and attracted Sphæres jointly, that is, as the Product of the Sphæres multiplied into each other, if the Distances of the Centers be the same. That is, the Weight of the Sphære B upon the Sphære A, is to the Weight of the Sphære D upon the Sphære C, as 27×7 to 8×6 , or as 189 to 48.

Corol. 4. AND if the Distances be unequal, Fig. 214. the Weights are as those Products directly, and the Squares of the Distances between the Centers inversely; that is, as $\frac{27 \times 7}{16}$ to $\frac{8 \times 6}{1}$, or as 11,81. to 48.

Corol. 5. THE same will hold good, if the Fig. 214. Attraction proceeds from both Sphæres exerting their attractive Force mutually upon each other; for then the Attraction proceeding from both Forces, will be twice as much, but the Proportion will remain the same as before. Thus (by Law 3.) Action and Reaction being equal, the Sphære B will attract the Sphære A, just as much as the Sphære A attracts the Sphære B; wherefore the whole mutual Attraction, or the whole Motion with which both of them tend towards each other will be $\frac{7 \times 27}{16} + \frac{27 \times 7}{16}$,

or

or $\frac{2 \times 7 \times 27}{2 \times 16}$, or 23, 62. And the whole Motion with which the Sphæres C and D attract each other, will be $\frac{8 \times 6}{1} + \frac{8 \times 6}{1}$, or $\frac{2 \times 8 \times 6}{1}$, or as 96, the same Proportion as before.

Corol. 6. IF any of these sort of Sphæres, revolve about other Sphæres that are at Rest, viz, a single Sphære about a single Sphære; and the Distances betwixt the Centers of the Sphæres that are at Rest, and the revolving Sphæres, be proportionable to the Diameters of the Sphæres which are at Rest; their periodical Times will be equal (*by Prop. IV. Cor. 3.*)

Corol. 7. AND on the contrary, if their periodical Times be equal, their Distances will be proportionable to the Diameters of the Sphæres which are at Rest, by the same *Corol.*

Corol. 8. ALL those things which are above demonstrated concerning the Motion of Bodies about the Focus's of Conick Sections, hold good here also, if the attracting Sphære be such a sort of one as is described in the Proposition, and if it be placed in the Focus of the Conick Section.

Corol. 9. THE same things will also hold good, if the revolving Sphæres be attractive ones also, and of the same sort as those described in the Proposition.

THE

THE foregoing Propositions not only explain to us the whole Nature of Gravitation here upon the Earth, but they also shew us how to find out and to compare together the Weights of Bodies upon different Planets, the Quantity of Matter in each Planet, and also the Densities of the several Planets, by *Prop. VIII. Theorem VIII* of the Third Book, which is this,

If the Matter of two Globes which gravitate towards each other mutually, be of the same kind in all places quite round them, which are at the same Distance from the Center; the Weight of either of the Globes upon the other, will be reciprocally as the Square of the Distances between their Centers.

AFTER I had found out that the Gravitation towards the whole Planet arose from, and was compounded of the Gravitation towards its Parts, and that it tended towards every one of those Parts in a reciprocal Proportion to the Square of the Distance from the Parts: I had some doubt whether this reciprocal duplicate Proportion, was exactly as the whole Force compounded of a great many Forces, or only very nearly. For it might have so happened that the Proportion, which at very great Distances, may be exact enough, might be considerably altered near the Superficies of the Planet, by Reason of the unequal Distances and dissimilar Situation of the Particles. But
at

length by *Prop. LXXV* and *LXXVI* of the first Book and their Corollaries, I found the Truth of this Proposition.

Corol. 1. HENCE the Weights of Bodies upon different Planets, may be found and compared together. For the Weights of equal (homogeneous) Bodies revolving about the Planets in Circles are as their centripetal Forces, that is (*by Corol. 2. Prop. IV.*) as the Diameters of the Circles directly, and the Squares of the periodical Times inversely; and their Weights upon the Superficies of the Planets, or at any other Distances from the Centers, are greater or less (*by this Proposition.*) in a duplicate Proportion of the Distances inversely. Thus

Days Hours

The periodical Time of
Venus about the Sun is 224—16 $\frac{3}{4}$

The periodical Time of
the outermost Satellity.
of *Jupiter* about Him. 16—16 $\frac{8}{15}$

The periodical Time
of *Huygens's* Satellity
about *Saturn*. 15—22 $\frac{2}{3}$

The periodical Time
of the Moon about
the Earth. 27—07—43 Minutes.

If these be compared with the mean Distance of *Venus* from the Sun, and with the greatest Heliocentrick Elongations of the outermost Satellity of *Jupiter* from the Center of *Jupiter*

ter, and of the *Huygenian* Satellyty from the Center of *Saturn*, and of the Moon from the Center of the Earth, the different Weights of equal Bodies at equal Distances from the Sun, *Jupiter*, *Saturn* and the Earth, will be found; and if these Distances be increased or diminished, the Weights will be diminished or increased in a duplicate Proportion; and consequently the Weights of equal Bodies upon the Superficies of the Sun, *Jupiter*, *Saturn* and the Earth, that is, at the Distances of their Semi-diameters, will be also found. Thus

The greatest Heliocentrick Elongation of the outermost Satellyty of

Jupiter from the Sun is '8---"16

Of the Satellyty of *Saturn* 3---04

Of the Moon from the Earth 10---33

Now the greatest Heliocentrick Elongation of a Satellyty from its Planet being known, that is, its Distance from the Center of the Planet, in Minutes and Seconds of the Planet's Orbit, the Proportion which this Distance bears to the Radius of that Orbit, that is, to the Distance of the Planet from the Sun, is known also by the common *Rule of Three*. Thus the mean Diameter of *Jupiter's* Orbit about the Sun, is to the Distance of the outermost Satellyty from the Center of *Jupiter*, as about 52012 to 124; And hence it appears by *Division*, that the mean Distance of *Venus* from the Center of the Sun, is to the Distance of

the same Satelity from *Jupiter*, as 7234 to 124. Now the periodical Times are $224\frac{2}{3}$ Days, and $16\frac{3}{4}$ Days. Wherefore if these Distances be divided by the Squares of the Times (by *Prop. IV. Cor. 2.*) the centripetal Force of the Satelity towards *Jupiter*, will be to the centripetal Force of *Venus* towards the Sun, as 442 to 143: But the Distance of the outermost Satelity from *Jupiter*, being to the Distance of *Venus* from the Sun, as 124 to 7234, the centripetal Force of the Satelity must be diminished as the Square of this Proportion, that the Distances may be the same; and then the centripetal Force of the Satelity towards *Jupiter*, if it were as far removed from it as *Venus* is from the Sun, would be to the centripetal Force of *Venus* towards the Sun, as $\frac{13}{100}$ to 143, or as 1 to 1100. In like manner, by computing most exactly, our Author found the Weights of equal Bodies at equal Distances from the Centers of the Sun, *Jupiter*, *Saturn* and the Earth to be as 1, $\frac{1}{1067}$, $\frac{1}{3021}$, and $\frac{1}{169282}$. And the Weight of equal Bodies at the Distances of 10000, 997, 791 and 109 from the Centers of the Sun, *Jupiter*, *Saturn* and the Earth, that is, on their Superficies, He found to be as 10000, 943, 529 and 435.

Carol.

Corol. 2. HENCE also the Quantity of Matter in each of the Planets is known, for the Quantity of Matter in every Planet is as its attractive Force at an equal Distance from its Center, that is, in the Sun 1, in *Jupiter*

$\frac{1}{1067}$, in *Saturn* $\frac{1}{3021}$, and in the Earth

$\frac{1}{169282}$.

Corol. 3. HENCE the Denfities of the Planets are also known. For the Weights of equal and homogeneous Bodies upon homogeneous Sphæres, are at the Superficies of those Sphæres, as the Diameters of the Sphæres (*by Prop. LXXII.*) and their Weights upon heterogeneous Sphæres, or Sphæres of different Denfities, must consequently (*by the Hypothesis.*) be in a compound Proportion of the Denfities and Diameters of the Sphæres, and therefore (*by Division.*) the Denfities will be as those Weights applied to the Diameters of the Sphæres, or the Denfities will be as the Weights of the equal homogeneous Bodies directly, and the Diameters of the Sphæres inversely. Now the true Diameters of the Sun, *Jupiter*, *Saturn*, and the Earth, are to each other as 10000, 997, 791 and 109, and the Weights of equal Bodies upon their Superficies, are as 10000, 943, 529 and 435; wherefore these latter Numbers divided by the former, will give the several Denfities, viz. 100, $94\frac{1}{7}$, 67 and 400.

X

THUS

THUS we have seen what the Result of the Composition of equal attractive Forces, decreasing as the Squares of the Distances from their Particles, will be in spherical Bodies mutually attracting each other. Let us now suppose a Sphære divided into an infinite Number of equal Particles, and that to every one of these Particles there tend equal centripetal Forces proportionable to the Distances from the Particles directly, and see what the Result of this Law will be, in the Tendency of several Sphæres towards each other.

PROP. LXXVII. THEOREM XXXVII.

If to every Point of any Sphæres, there tend centripetal Forces proportionable to the Distances of those Points from the attracted Bodies; I say, that the compound Force with which two Sphæres attract each other mutually, is as the Distance between the Centers of the Sphæres.

Case 1. LET AEBF be a Sphære; S its Center, P a Corpuscle attracted, PASB the Axis of the Sphære, passing through the Center the Corpuscle; EF, ef, two Planes cutting the Sphære perpendicular to this Axis, and equally distant from the Center of the Sphære on each side; Gg the Intersections of the Planes and the Axis, and let H be any Point in the Plane EF. The centripetal Force of the Point

Fig. 215.

Point H upon the Corpuscle P, according to the Direction of the Line HP is (*by the Hypothesis.*) as the Distance PH, and this Force (*by Corol. 2. of the Laws of Motion.*) may be resolved into two Forces, one according to the Direction HG, and porportionable to the Line HG; and the other according to the Direction PG, and proportionable to the Line GP; this latter is the Force of Attraction towards the Center S, and is as the Distance PG. By the same Rule, the Force of all the Points in the Plane EF, that is, the Force of the whole Plane, by which the Corpuscle P is attracted towards the Center S, is as the Distance PG, multiplied by all the Points of which the Plane EF is composed; that is, as the Product or Solid made up of the Plane EF, and the Distance PG. So likewise the Force of the Plane ef, by which the Corpuscle P is attracted towards the Center S, is as the Plane ef multiplied by the Distance Pg, or as the Plane EF, which is equal to the Plane ef, multiplied by the Distance Pg. And (*by Composition.*) the Sum of the Forces of both the Planes, is as the Plane EF, multiplied by the Sum of the Distances PG + Pg; that is, as the Plane EF multiplied by twice PS, the Distance of the Corpuscle from the Center of the Sphære; (because SG is equal to Sg.) or which is the same thing, as the Sum of the equal Planes EF + ef, multiplied by the same Distance PS. By the same way of Reasoning, the Forces of

all the Planes of which the whole Sphære is composed, and which are at equal Distances from the Center of the Sphære on each side, are as the Sum of all those Planes multiplied by the Distance PS , that is, in a Proportion compounded of the whole Sphære, and of the Distance PS , consequently if the Sphære be given, its attractive Force is as the Distance of the Corpuscle P from its Center, *viz.* PS . $W. W. D.$

Case 2. SUPPOSE now the Corpuscle P to attract the Sphære $AEBF$, and the Sphære not to attract the Corpuscle; it will follow by the same way of Reasoning, that the Force with which the whole Sphære is attracted by the Corpuscle, is as the Distance PS : For let the Sphære be distinguished into Planes as before; the Sums of every correspondent Point in the equidistant Planes on each side the Center, will be attracted by the Corpuscle P , in Proportion to the Distance from the Center PS .

Case 3. LET another Sphære P , be composed of an infinite Number of Corpuscles P ; and because the Force by which every one of these Corpuscles is attracted, is as the Distance of that Corpuscle from the Center of the first Sphære S , and consequently the Force is the same as if it came from the Center of that Sphære S ; therefore the whole Force with which all the Corpuscles contained in the second Sphære P , are attracted, that is, with which the whole Sphære P is attracted, will be

the same as if this Sphære were attracted by a Force which came from one single Corpuscle placed in the Center of the first Sphære S, and therefore is proportionable to the Distance of the Centers of the Sphæres.

Case 4. SUPPOSE now that both the Sphæres S and P do mutually attract each other, Fig. 216. then (by the Third Law of Motion, Action and Reaction being the same,) the Sphære S, will attract the Sphære P, just as much as the Sphære P attracts the Sphære S, and therefore the whole Force by which they both tend to each other, is double the Force with which the one of them tends towards the other, the foregoing Proportion still remaining the same.

Case 5. Now let the Corpuscle p , be placed within the Sphære AEBF; then because the Fig. 217. Force of the Plane ef upon the Corpuscle p , is as the Product of that Plane multiplied by the Distance pg ; and the Force of the Plane EF upon the same Corpuscle p , is as the Product of the Plane EF multiplied by the Distance pG ; but because the Corpuscle is attracted the contrary way by the Plane EF to what it is by the Plane ef , therefore the Force which arises from them both, must be as the Difference of the fore-mentioned Solids, or as $ef \times pg - EF \times pG$, that is, ef being equal to EF, and pg equal to $2Sp + pG$, therefore $ef \times pg - EF \times pG$ will be equal to $2EF \times Sp$, or dividing Sp by 2, the attractive Force is as $2EF \times Sp$, that is, as the Sum of the equal Planes

X 3

multi-

ef , EF , multiplied by half the Difference of the Distances gp , Gp ; or as that Sum multiplied by the Distance of the Corpuscle from the Center of the Sphære. And by the same way of Reasoning, the Attraction of all the Planes ef , EF , in the whole Sphære, that is, the Attraction of the whole Sphære, is as the Sum of all the Planes added together, or as the whole Sphære and the Distance of the Corpuscle p from the Center of the Sphære, viz. pS .

Fig. 217. *Case 6.* AND if we imagine another Sphære composed of an infinite Number of Particles p , and placed within the former Sphære $AEBF$; the same Argument as before, will hold for the Attractions being as the Distance of the Centers pS , whether we consider that Attraction singly in each Sphære upon the other, or upon both Sphæres mutually; for the Attractions of every Particle of each Sphære upon every Particle of the other Sphære, being as the Distance of the Centers of the Sphæres, it follows (*by Composition.*) that the Sum of the Attractions of all the Particles of which each Sphære is composed, must be in the same Proportion. W. W. D.

PROP. LXXVIII. THEOREM XXXVIII.

If there be Sphæres never so dissimilar and irregular as they go from the Center to the Circumference, but every where similar as they go along the Circumference at every given Distance

Distance from the Center ; and the attractive Force of every Point, be as its Distance from the attracted Body : I say, that the whole Force with which two such sort of Sphæres attract each other mutually, is proportionable to the Distance between the Centers of the Sphæres.

See the Figure belonging to *Prop. LXXVI.*

AND suppose, as in that Proposition, as many concentrick similar Sphæres as you please AB, CD, EF, so that the internal ones, added to or taken from the external ones, may make the Matter more or less dense towards the Center ; then (*by the foregoing Proposition.*) these Sphæres will attract as many other similar concentrick Sphæres as you please, that is, every single Sphære will attract every other single Sphære with Forces which are directly as the Distance between the Centers SP ; and therefore (*by Composition or Division.*) the Sum of all those Forces, or the Excess of some of them above the other ; that is, the Force with which the whole Sphære AB, composed of any Number of concentrick Sphæres, or of the Differences of concentrick Sphæres, attracts the whole Sphære GH, composed likewise of any Number of concentrick Sphæres, or of the Differences of concentrick Sphæres, will be in the same Proportion of the Distance SP. Let the Number of the concentrick Sphæres be increased infinitely, so that the Density of the

X 4

Matter,

Matter, and consequently the attractive Force as it goes from the Center to the Circumference, may increase or decrease, according to any Rule whatsoever, and let those Places which are less dense than the rest, be filled up with Matter which has no attractive Force, so that the Sphæres may be of what Form or Sort you please; and the Force with which the one attracts the other will still be, by the foregoing Argument, as the Distance between the Centers *SP* directly. W. W. D.

Corol. The things before demonstrated in the Xth and LXIVth Prop. concerning the Motion of Bodies about the Centers of Conick Sections, hold the same here also, where all the Attractions arise from the Force of such sort of sphærical Bodies as we have been now describing, and where the attracted Bodies are also of the same sort; for these being nothing else but Compositions of those simple Cases, the Proportions must continue the same by Multiplication or Division.

S C H O L I O N.

THUS we have explained two very remarkable Cases of Attraction. The first was, where the centripetal Forces decreased as the Squares of the Distances: and the second was, where the centripetal Forces increased directly as the Distances; and in both these Cases the Result is, that the sphærical Bodies themselves will

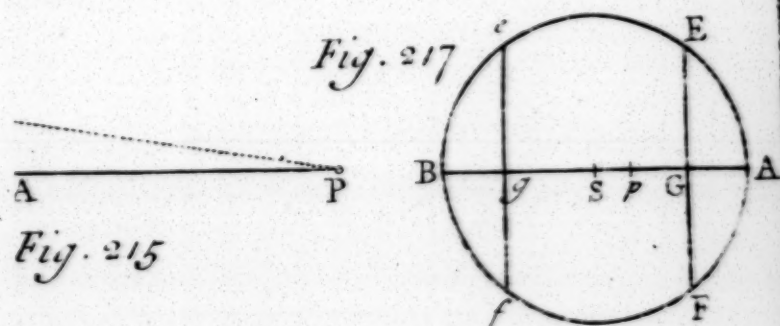
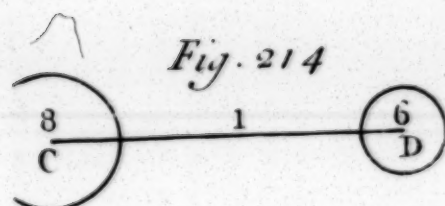
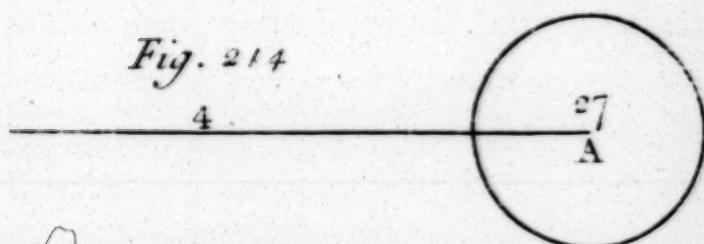
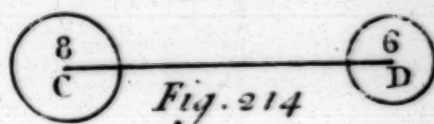
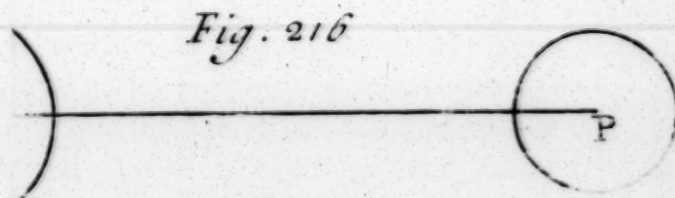


Fig. 215



s,
e
s,
y
ne
es
ill

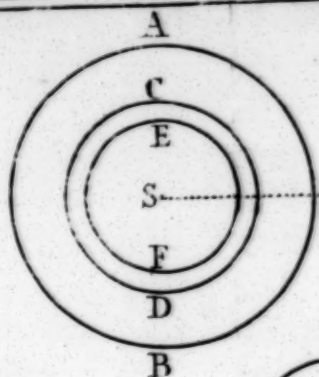


Fig. 213

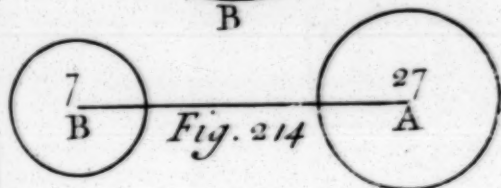
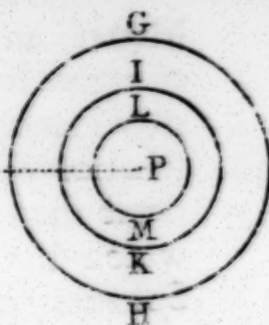


Fig. 214

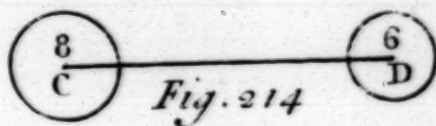


Fig. 214

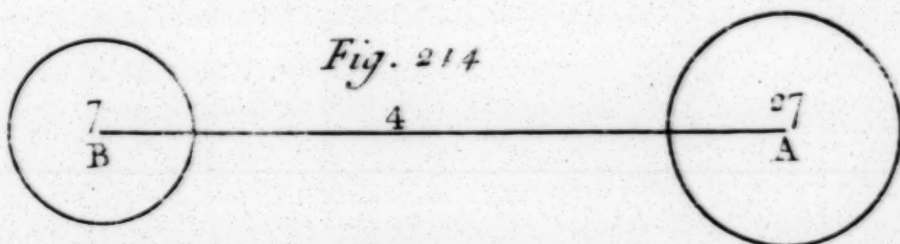


Fig. 214

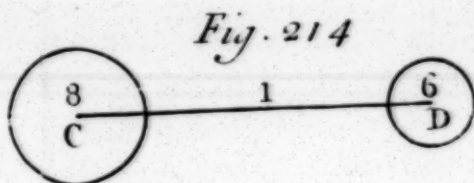


Fig. 214

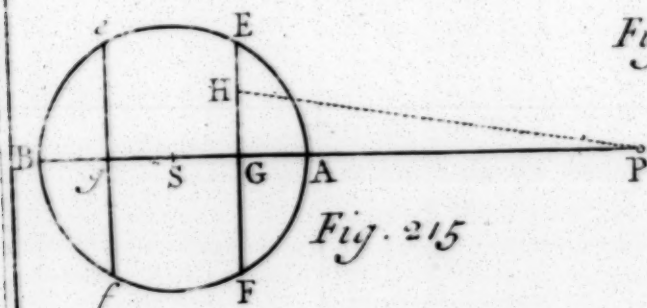


Fig. 215

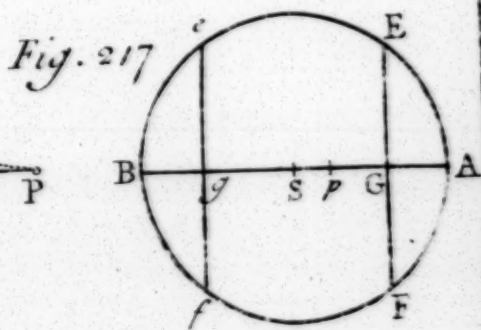


Fig. 217

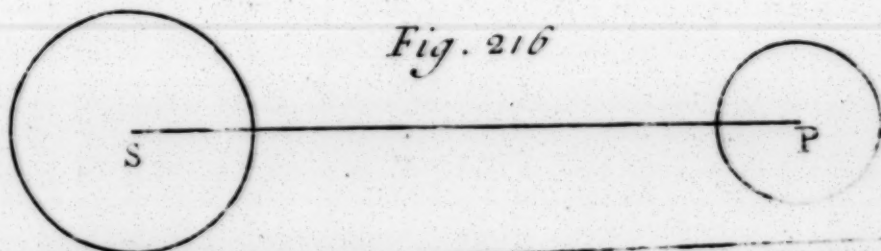
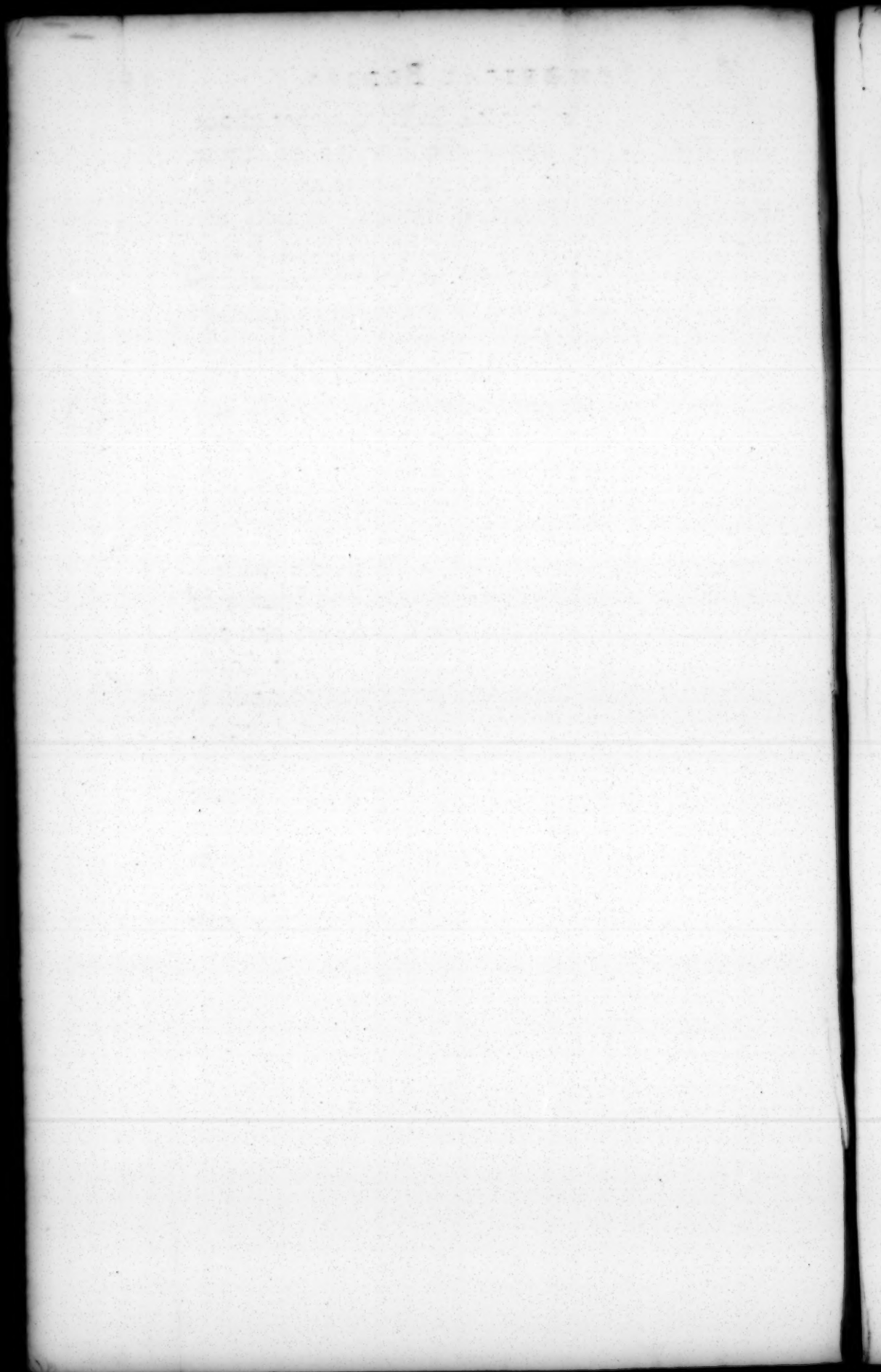
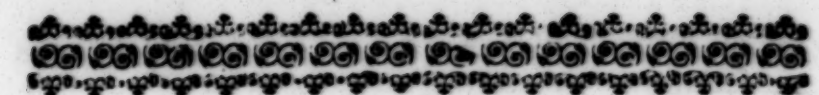


Fig. 216



will be moved in Conick Sections either about the Centers, or about the Focus's of those Sections; and also that the whole centripetal Forces of the sphærical Bodies, which are compounded of the Forces of all their Particles, increase or decrease as they are more remote from the Center, in the same Proportion that those centripetal Forces themselves do in their Distances from the single Particles; that is, if every single Particle, of which any Sphære is composed, attract a Corpuscle with a certain Force decreasing as the Square of the Distances between that Corpuscle and the Particles; the whole Sphære made up of such Particles will attract such a Corpuscle with a certain Force decreasing also as the Square of the Distances between the Corpuscle and the Center of the Sphære; and if every single Particle of which another Sphære is composed, attract a Corpuscle with a certain Force which increases directly as the Distances between the Particle and the Corpuscle; the whole Sphære made up of such Particles will attract such a Corpuscle with a certain Force which increases as the Distances between the Corpuscle and the Center of the Sphære directly, which is a very observable Thing.

F I N I S.



THE INDEX.

A.

- A**CTION and Reaction always equal, and contrary to each other. Page 29
- Angle contained between a Chord and a Tangent, is at last a vanishing Angle. 80
- Angles of Contact, not all equal, but some infinitely less than others. 94
- Arch of a Body revolving in a Circle compared with its direct Descent. 124
- Area's, described by Bodies moving in any Orb, are proportionable to the Times of Description. 105
- Attraction, what is meant by it. 277
- Bodies mutually attracting each other, describe similar Figures about their common Center of Gravity, and about each other. 262
- These Figures not alter'd, if one of the Bodies be made to rest, and the other to move about it. 265
- The periodical Times of such revolving Bodies computed. 270
- As. What is meant by one Quantity being as another Quantity. 87

B.

- Balance, the Explication of it. 32, 59
- Body or Mass, what meant by it. 2

C.

- Centrifugal Force compared with the Force of Gravity. 128
- Centri-

The INDEX.

<i>Centripetal Force, how to measure it.</i>	Page 108
—— <i>How to compare it with the Force of Gravity.</i>	ibid.
—— <i>How to discover it.</i>	109
—— <i>If it decreases as the Squares of the Distances, it will make Bodies revolve in some Conick Section.</i>	181
<i>Circles, the centripetal Forces of Bodies moving in them computed.</i>	117
—— <i>What Proportions they bear to each other.</i>	152
—— <i>These Forces compared with the same Forces in other Curves.</i>	155
<i>Curvature, of Curves, how to find it in any Ellipsis.</i>	163

D.

<i>Descent of heavy Bodies, what Proportion it bears to the Time.</i>	46
—— <i>How to determine the Spaces which they descend through in given Times.</i>	205
—— <i>Instances in the Planets.</i>	209
—— <i>How to determine the Times of such Descent.</i>	220,
	221
—— <i>The Velocities, Spaces and Times determined, upon Supposition that the centripetal Force is directly as the Distance.</i>	223
—— <i>The same applied to Bodies descending and ascending within the Surface of the Earth.</i>	226

E.

<i>Earth, all its Parts press equally upon each other.</i>	57
—— <i>Its true Figure determined.</i>	259
<i>Ellipsis, what the Law of centripetal Force is to make a Body move about the Center of it.</i>	161
<i>Engines, a general Rule for explaining their Powers.</i>	93
—— <i>What the Law of centripetal Force is, to make a Body move about the Focus of it.</i>	169

F.

<i>Figures (similar) proportionable to their respective sides</i>	
—— <i>Their Area's, what Proportion they bear to their sides.</i>	79
—— <i>Inscribed and circumscribed become at last equal.</i>	76
	Force

The INDEX.

<i>Force, the inherent Force of Matter, what meant by it.</i>	3
<i>Impressed Force, what meant by it.</i>	5
<i>Centripetal Force, what meant by it.</i>	6
<i>Centripetal Force (absolute) what and how to measure the Quantity of it.</i>	8
<i>Centripetal Force (accelerative) what and how to measure the Quantity of it.</i>	9
<i>Centripetal Force (Motive) what and how to measure the Quantity of it.</i>	10
<i>Of Compound Forces.</i>	30

G.

<i>Gravity, the Nature of it explained.</i>	69, 125
— <i>Its common Center not alter'd by the Action of Bodies upon each other.</i>	41
— <i>Its Action mutual in all the Planets.</i>	144
— <i>Its different Effects upon different Suppositions</i>	214
— <i>Its common Center of the whole System, where to be placed.</i>	275

H.

<i>Hyperbola, what the Law of centripetal Force is to make a Body move in one about the Focus.</i>	173
<i>Hypothesis, the Copernican the true one.</i>	115

I.

<i>Inclined Plane, the Motion of Bodies upon it explained.</i>	60
--	----

L.

<i>Leaver, its Force explained.</i>	92
-------------------------------------	----

M.

<i>Matter, how to measure the Quantity of it.</i>	1
— <i>Its universal Properties.</i>	102
— <i>All its Parts gravitate towards each other.</i>	103
<i>Moon, kept in its Orb by gravitating towards the Earth.</i>	139
<i>Motion, (absolute) what meant by it.</i>	14
<i>Motion (relative) what meant by it.</i>	ibid.
— <i>How</i>	

The INDEX.

— How to distinguish absolute Motion from relative.	21
— Circular, how to be computed.	24
— Its Alteration always proportionable to the impressed Force.	28
— How to measure it in general.	2
— Its Quantity not altered by the Action of Bodies upon each other.	36
Motion of Bodies in a given Space.	7
The same, whether that Space be at Rest or no.	45
Motion of Bodies not altered by being moved also in parallel Lines.	46
— How to represent it geometrically.	67
O.	
Orbs, of the five primary Planets surround the Sun.	135
— How to measure the centripetal Force of a Body moving in any.	147, 148, 149
— Their Latus rectum's compared with their Area's.	185
— How to find that which a Body moves in, from the Velocity given in the principal Vertex.	202
P.	
Parabola, what the Law of centripetal Force is to make a Body move in one about the Focus	179
Pendulums, how to make use of them.	50
— How to make them vibrate in given Cycloids.	243
— Their Times of Vibration all equal, if the centripetal Force be as the Distance from the Center directly.	245
— How to determine the Velocities and Times of Vibrations.	240
— How to compare the Times of vibrating, revolving and falling Bodies with each other.	255
— Their Usefulness in Astronomy.	258
Place, (absolute) what meant by it.	14
— (Relative) what meant by it.	14
Planets, their Motion about the Sun explained.	70
— Their periodical Times compared with their Distances.	136
— How	

The INDEX.

— <i>How to compute the Times of their direct Fall into the Sun.</i>	209
<i>Projected Bodies, their Motion in a Parabola.</i>	46, 48.
<i>Proportions, the nature of first and last Proportions.</i>	65, 66, &c.
<i>Pulley, its Motion explained.</i>	60
Q.	
<i>Quantities, generated by local Motion.</i>	68
— <i>Continually decreasing in a given Time, become equal at last.</i>	73, 74, &c.
— <i>Of the same kind, proportionable to their parts, if divided into the same number of parts.</i>	78
R.	
<i>Rest (real) what meant by it.</i>	15
— <i>(Relative) what meant by it.</i>	ib.
S.	
<i>Satellites, of Jupiter, their periodical Times and their Distances.</i>	131
— <i>Of Saturn, their periodical Times and Distances.</i>	133
— <i>The Forces by which they are kept in their Orbs compared with their Distances.</i>	138
— <i>Kept in their Orbs by the Force of Gravity.</i>	143
<i>Screw, how to compute its Force.</i>	35
<i>Semicircle, what the Law of centripetal Force is, to make a Body move in one.</i>	157
<i>Space (absolute) what meant by it.</i>	13
— <i>(Relative) what meant by it.</i>	ibid.
<i>Spaces, what their Proportions in falling Bodies at the beginning.</i>	84
<i>Spheres, their attractive Forces.</i>	287, &c.
<i>Spherical Superficies, their attractive Forces.</i>	281, 282
<i>Spherical Bodies, their attractive Forces.</i>	279
<i>Spiral Line, what the Law of centripetal Force is, to make a Body move in one.</i>	159
<i>System, the present System of the World explained by Gravity.</i>	183
	— 111

The INDEX.

—*Its Beauty and Simplicity.* 273

T.

- Time, (absolute) what meant by it.* 13
 —(*Relative*) *what meant by it.* *ibid.*
 —*How to represent it geometrically.* 66
 —*Periodical Time of Bodies moving in Circles, how computed.* 119, 120, &c.
 —*Periodical Time of Bodies moving in Ellipses, about their Centers.* 164
 —*The same demonstrated in other Figures.* 168
 —*Periodical Time compared with the longer Axis's in Ellipses.* 187
 —*How to determine the Time of a Body's direct Descent, the centripetal Force being according to any Law.* 227

V.

- Vanishing Arches, Chords and Tangents, all equal.* 80
Vanishing Triangles, similar, and their last Proportion equal. 82
 —*Their Areas's in a duplicate Proportion to their correspondent sides.* 83
Vanishing Subtense of the Angle of Contact, what Proportion it bears to its conterminat Arch. 89
Velocity, how geometrically represented. 67
 —*Projectile, how measured.* 107
 —*In Circles.* 119
 —*Of Bodies moving in Ellipses.* 188
 —*Of Bodies moving in Ellipses, compared with that of Bodies moving in Circles.* 190
 —*Of Bodies moving in a Parabola or Hyperbola.* 194
 —*These Velocities compared with the Velocities of Bodies moving in Circles.* 195
 —*Of a Body moving in any Conick Section, compared with the Velocity of a Body moving in any Circle.* 197
 —*Of a Body falling directly, compared with its Velocity in a Circle at the same Distance from the Center.* 211
 —*Of*

The INDEX.

—Of a Body falling in a Parabola, compared with its Velocity in a Circle at the same Distance. 215

—How determined in falling Bodies, the centripetal Force being according to any Law. 227

W.

Wedge, how to compute its Force. 35, 92.

Weights, of Bodies on different Planets, how found, and how compared with each other. 302

Windless, its Power explained. 60

E R R A T A.

Page	Line	for	read
32	24	to it and perpendicular- lar to DC	and DC perpendicu- lar to it.
41	26	Fig. 6.	Fig. 5 and 6.
61	5	Hand A	Hand H.
81	23	Fig. 27.	Fig. 26.
84	8	$AE \times E$	$AE \times EC$
92	4 & 24	Fig. 33.	Fig. 31.
		$\frac{V}{R} = \frac{v}{r}$	$\frac{V}{R} = \frac{v}{r}$
119	21	$\frac{R}{r}$	$\frac{R}{r}$
145	16	QBRC	QB, RC.
160	3	multiplied	multiplied by
167	19	Axis ab	Axis ac
172	13	cP	CP
185	15	Ellipsis QP	Ellipsis ABCD
185	17	Ellipsis qr	Ellipsis $abcd$
192	7	QPT	YPT
193	10	Semicle	Semicircle



